

Partial differential equations

Definition:- An equation involving one or more partial derivatives of a function of two or more independent variables is known as a partial differential equation (PDE).

* The highest derivative gives the order of PDE and the positive integral power of highest derivative gives the degree of PDE.

Note:- let z be a function of two independent variables x, y .

The following are the standard notations.

a) $p = \frac{\partial z}{\partial x} = z_x$ b) $q = \frac{\partial z}{\partial y} = z_y$ c) $r = \frac{\partial^2 z}{\partial x^2} = z_{xx}$

d) $s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = z_{xy} = z_{yx}$ e) $t = \frac{\partial^2 z}{\partial y^2} = z_{yy}$

Formation of PDE by eliminating arbitrary constants:-

Consider the function $f(x, y, z, a, b) = 0$ where z is a function of x and y with a, b arbitrary constants.

The PDE for the function 'f' is formed by differentiating 'f' w.r.t x and y and eliminating arbitrary constants.

* If number of arbitrary constants exceeds the independent variables, then it is necessary to find higher derivatives to eliminate arbitrary constants and form PDE.

Form PDE by eliminating arbitrary constants in the following functions :-

① $z = (x-a)^2 + (y-b)^2$

soln: given $z = (x-a)^2 + (y-b)^2$ — (1)

diff (1) wrt 'x' part

$$\frac{\partial z}{\partial x} = p = 2(x-a) \text{ — (2)}$$

diff (1) wrt 'y' part

$$\frac{\partial z}{\partial y} = q = 2(y-b) \text{ — (3)}$$

using (2) & (3) in (1), we get

$$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$$\Rightarrow \boxed{4z = p^2 + q^2} \rightarrow \text{Required PDE.}$$

② $dz = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

soln: given $dz = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ — (1)

diff (1) wrt 'x' part

$$dp = \frac{2x}{a^2} \Rightarrow p = \frac{x}{a^2} \text{ — (2)}$$

diff (1) wrt 'y' part

$$dq = \frac{2y}{b^2} \Rightarrow q = \frac{y}{b^2} \text{ — (3)}$$

using (2) & (3) in (1), we get

$$\boxed{dz = p dx + q dy} \rightarrow \text{Required PDE.}$$

$$\textcircled{3} \quad z = a \log(x^2 + y^2) + b$$

soly given $z = a \log(x^2 + y^2) + b$ — (1)

diff (1) wrt 'x' par

$$p = \frac{\partial z}{\partial x} = \frac{2ax}{x^2 + y^2} \quad \text{--- (2)}$$

diff (1) wrt 'y' par

$$q = \frac{\partial z}{\partial y} = \frac{2ay}{x^2 + y^2} \quad \text{--- (3)}$$

$$\frac{\textcircled{2}}{\textcircled{3}} \Rightarrow \frac{p}{q} = \frac{x}{y} \Rightarrow \boxed{py - qx = 0} \rightarrow \text{Required PDE.}$$

$$\textcircled{4} \quad z = xy + y\sqrt{x^2 - a^2} + b$$

soly given $z = xy + y\sqrt{x^2 - a^2} + b$ — (1)

diff (1) wrt 'x' par

$$p = y + \frac{xy}{\sqrt{x^2 - a^2}} \quad \text{--- (2)}$$

diff (1) wrt 'y' par

$$q = x + \sqrt{x^2 - a^2} \Rightarrow \sqrt{x^2 - a^2} = q - x \quad \text{--- (3)}$$

using (3) in (2), we get

$$p = y + \frac{xy}{q - x}$$

$$\Rightarrow (p - y)(q - x) = xy$$

$$pq - px - qy + xy = xy$$

$$\Rightarrow \boxed{pq = px + qy} \rightarrow \text{Required PDE.}$$

$$(5) \quad z = a \log \left(\frac{b(y-1)}{1-x} \right)$$

soly: Given $z = a \log \left(\frac{b(y-1)}{1-x} \right) \quad \text{--- (1)}$

diff (1) wrt 'x' part

$$P = \frac{a(1-x)}{b(y-1)} \times \frac{-b(y-1)(-1)}{(1-x)^2}$$

$$\Rightarrow P = \frac{+a}{1-x} \quad \text{--- (2)}$$

diff (1) wrt 'y' part

$$q = \frac{a(1-x)}{b(y-1)} \times \frac{b}{(1-x)}$$

$$\Rightarrow q = \frac{a}{y-1} \quad \text{--- (3)}$$

$$\frac{(2)}{(3)} \Rightarrow \frac{P}{q} = \frac{+(y-1)}{(1-x)} \Rightarrow \boxed{P(1-x) = q(y-1)} \rightarrow \text{Required PDE}$$

$$(6) \quad z = ax + by + cxy$$

soly: Given $z = ax + by + cxy \quad \text{--- (1)}$

diff (1) wrt 'x' part

$$P = a + cy \quad \text{--- (2)}$$

diff (1) wrt 'y' part

$$q = b + cx \quad \text{--- (3)}$$

diff (2) wrt 'x' part

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = r = 0 \quad \text{--- (A)}$$

diff (5) wrt 'y' par

$$\frac{\partial^2 z}{\partial y^2} = t = 0 \quad \text{--- (5)}$$

diff (1) wrt 'y' par

$$\frac{\partial z}{\partial y \partial x} = s = c \quad \text{--- (6)}$$

using (2), (3) and (6) in (1) we get

$$z = (p - sy)x + (q - sx)y + sxy$$

$$z = px - sxy + qy - sxy + sxy$$

$$\Rightarrow \boxed{z = px + qy - sxy} \Rightarrow \text{Required PDE.}$$

$$(7) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

solve given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{--- (1)}$

diff (1) wrt 'x' par

$$\frac{\partial x}{\partial a^2} + \frac{\partial z p}{\partial c^2} = 0 \Rightarrow \frac{x}{a^2} + \frac{z p}{c^2} = 0 \quad \text{--- (2)}$$

diff (1) wrt 'y' par

$$\frac{\partial y}{\partial b^2} + \frac{\partial z q}{\partial c^2} = 0 \Rightarrow \frac{y}{b^2} + \frac{z q}{c^2} = 0 \quad \text{--- (3)}$$

diff (2) wrt 'x' par

$$\frac{1}{a^2} + \frac{1}{c^2} [z r + p^2] = 0 \Rightarrow \frac{1}{a^2} = - \frac{[z r + p^2]}{c^2} \quad \text{--- (4)}$$

using (4) in (2), we get

$$-x \frac{[z r + p^2]}{c^2} + \frac{z p}{c^2} = 0$$

$$\Rightarrow \nabla x [z \nabla + P'] = \nabla z P$$

$$\Rightarrow \boxed{z P = z \nabla x + x P^2} \rightarrow \text{Required PDE.}$$

Formation of PDE by eliminating arbitrary functions:-

Consider z , a function of x and y containing arbitrary functions. By differentiating z partially w.r.t x and y for an appropriate number of times, it is possible to eliminate the arbitrary functions, the resulting equation will be a PDE.

* If z is containing a single arbitrary function, then possible first derivatives are sufficient to eliminate arbitrary functions. If z is containing two arbitrary functions, the possible second derivatives are sufficient to eliminate arbitrary functions.

Form PDE by eliminating arbitrary functions in the following functions:-

$$\textcircled{1} \quad z = f(x^2 - y^2)$$

soln: Given $z = f(x^2 - y^2)$ — (1)

diff (1) wrt 'x' part

$$P = f'(x^2 - y^2) \cdot 2x \quad \text{--- (2)}$$

diff (1) wrt 'y' part

$$Q = f'(x^2 - y^2) \cdot (-2y) \quad \text{--- (3)}$$

$$\frac{p}{q} = \frac{x}{-y} \Rightarrow \boxed{py + qx = 0} \rightarrow \text{Required PDE.}$$

$$\textcircled{2} \quad z = e^{my} \phi(x-y)$$

soly. given $z = e^{my} \phi(x-y) - \textcircled{1}$

diff $\textcircled{1}$ wrt 'x' par

$$p = e^{my} \phi'(x-y) - \textcircled{2}$$

diff $\textcircled{1}$ wrt 'y' par

$$q = m e^{my} \phi(x-y) - e^{my} \phi'(x-y) - \textcircled{3}$$

using $\textcircled{1}$ & $\textcircled{2}$ in $\textcircled{3}$, we get

$$q = mz - p \Rightarrow \boxed{p + q = mz} \rightarrow \text{Required PDE.}$$

$$\textcircled{3} \quad z = y^2 + \int f\left(\frac{1}{x} + \log y\right)$$

soln given $z = y^2 + \int f\left(\frac{1}{x} + \log y\right) - \textcircled{1}$

diff $\textcircled{1}$ wrt 'x' par

$$p = -\frac{d}{dx} f\left(\frac{1}{x} + \log y\right) - \textcircled{2}$$

diff $\textcircled{1}$ wrt 'y' par

$$q = 2y + \frac{d}{dy} f\left(\frac{1}{x} + \log y\right) - \textcircled{3}$$

using $\textcircled{2}$ in $\textcircled{3}$ & simplifying we get

$$\textcircled{3} \Rightarrow qy = 2y^2 - px^2$$

$$\Rightarrow \boxed{px^2 + qy = 2y^2} \rightarrow \text{Required PDE.}$$

$$(4) \quad z = e^{ax+by} f(ax-by)$$

soln: Given $z = e^{ax+by} f(ax-by) \quad \text{--- (1)}$

diff (1) wrt 'x' part

$$p = a e^{ax+by} f(ax-by) + a e^{ax+by} f'(ax-by) \quad \text{--- (2)}$$

diff (1) wrt 'y' part

$$q = b e^{ax+by} f(ax-by) - b e^{ax+by} f'(ax-by) \quad \text{--- (3)}$$

consider, $b(2) + a(3)$

$$\Rightarrow bp + aq = ab e^{ax+by} f(ax-by) + ab e^{ax+by} f'(ax-by) + ab e^{ax+by} f(ax-by) - ab e^{ax+by} f'(ax-by)$$

$$\Rightarrow bp + aq = 2ab e^{ax+by} f(ax-by)$$

using (1), we get

$$\boxed{bp + aq = 2abz} \rightarrow \text{Required PDE.}$$

$$(5) \quad z = y f(x) + x \phi(y)$$

soln: Given $z = y f(x) + x \phi(y) \quad \text{--- (1)}$

diff (1) wrt 'x' part

$$p = y f'(x) + \phi(y) \quad \text{--- (2)}$$

diff (1) wrt 'y' part

$$q = f(x) + x \phi'(y) \quad \text{--- (3)}$$

diff (2) wrt 'x' part

$$r = y f''(x) \quad \text{--- (4)}$$

diff (3) wrt 'y' part

$$t = x \phi''(y) \quad \text{--- (5)}$$

diff (2) wrt 'y' par

$$s = f'(x) + \phi'(y) \quad \text{--- (6)}$$

consider x (2) + y (3)

$$\Rightarrow px + qy = xy f'(x) + x \phi(y) + y f(x) + xy \phi'(y)$$

$$px + qy = xy (f'(x) + \phi'(y)) + x \phi(y) + y f(x)$$

using (1) & (6)

$$\Rightarrow \boxed{px + qy = xy s + z} \rightarrow \text{Required PDE.}$$

(7)

$$z = f_1(y - 2x) + f_2(2y - x)$$

soln:

$$\text{Given } z = f_1(y - 2x) + f_2(2y - x) \quad \text{--- (1)}$$

diff (1) wrt 'x' par

$$p = -2 f_1'(y - 2x) - f_2'(2y - x) \quad \text{--- (2)}$$

diff (1) wrt 'y' par

$$q = f_1'(y - 2x) + 2 f_2'(2y - x) \quad \text{--- (3)}$$

diff (2) wrt 'x' par

$$r = 4 f_1''(y - 2x) + f_2''(2y - x) \quad \text{--- (4)}$$

diff (3) wrt 'y' par

$$t = f_1''(y - 2x) + 4 f_2''(2y - x) \quad \text{--- (5)}$$

diff (2) wrt 'y' par

$$s = -2 f_1''(y - 2x) - 2 f_2''(2y - x) \quad \text{--- (6)}$$

consider $2 \times$ (6) + (4)

$$\Rightarrow 2s + r = -3 f_2''(2y - x) \quad \text{--- (7)}$$

consider $\partial x(5) + (6)$

$$\Rightarrow \partial t + \delta = \delta f_z'' (\partial y - \alpha) - (8)$$

$$\frac{(7)}{(8)} \Rightarrow \frac{\partial \delta + \lambda}{\partial t + \delta} = \frac{-3}{6}$$

$$\Rightarrow 4\delta + \partial \lambda = -\partial t - \delta$$

$$\Rightarrow \boxed{\partial \lambda + \partial t + 5\delta = 0} \rightarrow \text{Required PDE.}$$

$$(7) \quad \phi(x+y+z, xy+z^2) = 0$$

Soly: let the given function be $\phi(u, v) = 0 - (1)$

$$\text{where } u = x+y+z, \quad v = xy+z^2$$

diff (1) wrt 'x' par,

$$\frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial v} \frac{\partial v}{\partial x} - (2)$$

diff (1) wrt 'y' par

$$\frac{\partial \phi}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial y} = -\frac{\partial \phi}{\partial v} \frac{\partial v}{\partial y} - (3)$$

$$\text{consider } \frac{(2)}{(3)} \Rightarrow \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = \frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right) \left(\frac{\partial v}{\partial y}\right) - \left(\frac{\partial v}{\partial x}\right) \left(\frac{\partial u}{\partial y}\right) = 0 - (4)$$

$$\text{we have } u = x+y+z$$

$$v = xy+z^2$$

$$\frac{\partial u}{\partial x} = 1+p$$

$$\frac{\partial v}{\partial x} = y+2zp$$

$$\frac{\partial u}{\partial y} = q+1$$

$$\frac{\partial v}{\partial y} = x + \alpha z q$$

substituting above derivatives in (4)

$$(4) \Rightarrow (p+1)(x + \alpha z q) - (q+1)(y + \alpha z p) = 0$$

$$px + x + \alpha z pq + \alpha z q - qy - y - \alpha z pq - \alpha z p = 0$$

$$\Rightarrow \boxed{x - y + px - qy + \alpha z(q - p) = 0} \rightarrow \text{Required PDE.}$$

$$(8) \quad f\left(\frac{xy}{z}, z\right) = 0$$

soly.

$$\text{Given function be } f(u, v) = 0 \quad (1)$$

$$\text{where } u = \frac{xy}{z} \quad v = z$$

diff (1) wrt 'x' par

$$\frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} = - \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} \quad (2)$$

diff (1) wrt 'y' par

$$\frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} = - \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} \quad (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = \frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right) \left(\frac{\partial v}{\partial y}\right) - \left(\frac{\partial u}{\partial y}\right) \left(\frac{\partial v}{\partial x}\right) = 0 \quad (4)$$

$$\text{we have } u = \frac{xy}{z} \quad v = z$$

$$\frac{\partial u}{\partial x} = y \left(\frac{z - xp}{z^2}\right) \quad \frac{\partial v}{\partial x} = p$$

$$\frac{\partial u}{\partial y} = x \left(\frac{z - yq}{z^2}\right) \quad \frac{\partial v}{\partial y} = q$$

substituting above derivatives in (4)

$$(4) \Rightarrow y \left(\frac{z-xp}{z^2} \right) q - x \left(\frac{z-yq}{z^2} \right) p = 0$$

$$\Rightarrow zyq - xy pq - xzp + xy pq = 0$$

$$\Rightarrow z(qy - px) = 0$$

$$\Rightarrow \boxed{qy - px = 0} \rightarrow \text{Required PDE.}$$

Note:-

A PDE is said to be homogeneous, if each and every term is either the dependent variable or its derivatives. Otherwise, PDE is said to be Non-homogeneous.

eg (1) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u$ - homogeneous

(2) $\frac{\partial u}{\partial x} + x = y$ - Non-homogeneous

(3) $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial x \partial y}$ - homogeneous

(4) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + z = \frac{\partial z}{\partial x} + xy$ - Non-homogeneous.

Solution of Non-homogeneous PDE:-

The solution of Non-homogeneous PDE is got by ^{performing} direct integration, taking the integration constants as arbitrary functions.

Problems:-

① solve $\frac{\partial^2 u}{\partial x^2} = x + y$

soln: Given $\frac{\partial^2 u}{\partial x^2} = x + y$ — ①

It is Non-homogeneous PDE.

integrate ① wrt 'x', treating 'y' constant.

$$\frac{\partial u}{\partial x} = \frac{x^2}{2} + xy + f(y)$$

integrate wrt 'x', treating 'y' constant.

$$u = \frac{x^3}{6} + \frac{x^2 y}{2} + x f(y) + g(y) \rightarrow \text{Required Soln}$$

② Solve $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$

soln: Given $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$

integrate wrt 'x', treating 'y' constant

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\sin(2x + 3y)}{2} + f(y)$$

integrate wrt 'x', treating 'y' constant

$$\frac{\partial z}{\partial y} = -\frac{\cos(2x + 3y)}{4} + x f(y) + g(y)$$

integrate wrt 'y', treating 'x' constant

$$z = -\frac{\sin(2x + 3y)}{12} + x F(y) + \phi(y) + h(x)$$

Required soln. ←

where $F(y) = \int f(y) dy$

$\phi(y) = \int g(y) dy$

③ Solve $\frac{\partial^2 u}{\partial x \partial t} = \sin x \sin t$, given $\frac{\partial u}{\partial t} = -2 \sin t$ when $x=0$, and $u=0$, when t is odd multiple of $\frac{\pi}{2}$.

soln. Given $\frac{\partial^2 u}{\partial x \partial t} = \sin x \sin t$ — (1)

integrate wrt 'x', treating 't' constant

$$\frac{\partial u}{\partial t} = -\cos x \sin t + f(t) \text{ — (2)}$$

integrate wrt 't', treating 'x' constant

$$u = \cos x \cos t + F(t) + g(x) \text{ — (3)}$$

$$\text{where } F(t) = \int f(t) dt.$$

put $x=0$ in (2)

$$\text{(2)} \Rightarrow -2 \sin t = -\sin t + f(t)$$

$$\Rightarrow \boxed{f(t) = -\sin t}$$

$$\Rightarrow F(t) = \int -\sin t dt = \cos t$$

$$\boxed{F(t) = \cos t}$$

put $t = (2n+1)\frac{\pi}{2}$ in (3)

$$\text{(3)} \Rightarrow 0 = \cos x \cos t + \cos(2n+1)\frac{\pi}{2} + g(x)$$

$$\Rightarrow \boxed{g(x) = 0}$$

substituting the arbitrary function values in (3)

$$\Rightarrow \underline{u = \cos x \cos t + \cos t} \rightarrow \text{Required soln}$$

④ Solve $\frac{\partial^2 u}{\partial y \partial x} = e^{-y} \cos x$, given $u=0$, when $y=0$ and

$\frac{\partial u}{\partial y} = 0$, when $x=0$. Also, s.t $u \rightarrow \sin x$ as $y \rightarrow \infty$

solve

$$\text{Given } \frac{\partial^2 u}{\partial y \partial x} = e^{-y} \cos x$$

$$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = e^{-y} \cos x \quad \text{--- (1)}$$

integrate wrt 'x' treating 'y' constant

$$\frac{\partial u}{\partial y} = e^{-y} \sin x + f(y) \quad \text{--- (2)}$$

integrate wrt 'y' treating 'x' constant.

$$u = -e^{-y} \sin x + F(y) + g(x) \quad \text{--- (3)}$$

where $F(y) = \int f(y) dy$

put $x=0$ in (3)

$$\text{(3)} \Rightarrow 0 = 0 + f(y) \Rightarrow \boxed{f(y) = 0}$$

$$F(y) = \int 0 dy = 0 \text{ or constant}$$

$$\text{we take } \boxed{F(y) = 0}$$

put $y=0$ in (3)

$$\text{(3)} \Rightarrow 0 = -\sin x + 0 + g(x)$$

$$\Rightarrow \boxed{g(x) = \sin x}$$

substituting arbitrary functions in (3)

$$\text{(3)} \Rightarrow u = -e^{-y} \sin x + \sin x$$

$$u = \underline{\underline{\sin x (1 - e^{-y})}} \rightarrow \text{Required soln}$$

--- (4)

put $y \rightarrow \infty$ in (4)

$$\text{(4)} \Rightarrow u = \sin x (1 - e^{-\infty}) \Rightarrow u = \sin x$$

\therefore as $y \rightarrow \infty$, $u \rightarrow \sin x$.
Hence proved

(5) Solve $\frac{\partial^2 z}{\partial x \partial y} = x^2 y$ given $z(x, 0) = x^2$ and $\frac{\partial z}{\partial y}(1, y) = \cos y$.

sol. Given $\frac{\partial^2 z}{\partial x \partial y} = x^2 y$ — (1)

integrate wrt 'x', treating 'y' constant

$$\frac{\partial z}{\partial y} = \frac{x^3 y}{3} + f(y) \text{ — (2)}$$

integrate wrt 'y', treating 'x' constant

$$z = \frac{x^3 y^2}{6} + F(y) + g(x) \text{ — (3)}$$

where $F(y) = \int f(y) dy$.

put $x=1$ in (1)

$$(2) \Rightarrow \cos y = \frac{y}{3} + f(y) \Rightarrow \boxed{f(y) = \cos y - \frac{y}{3}}$$

$$F(y) = \int \left(\cos y - \frac{y}{3} \right) dy$$

$$\boxed{F(y) = \sin y - \frac{y^2}{6}}$$

put $y=0$ in (3)

$$(3) \Rightarrow x^2 = 0 + 0 + g(x) \Rightarrow \boxed{g(x) = x^2}$$

substituting arbitrary function values in (3)

$$(3) \Rightarrow z = \frac{x^3 y^2}{6} + \sin y - \frac{y^2}{6} + x^2 \rightarrow \text{Required sol.}$$

easy.

Solution of Homogeneous equation involving the derivative w.r.t one independent variable only

The solution of homogeneous equation involving the derivative w.r.t one independent variable only is got by first treating the PDE as ODE and finding solution of ODE. The arbitrary constants in the solution of ODE are replaced by arbitrary functions of other independent variable giving a solution of the PDE.

Problems:-

① solve $\frac{\partial^2 z}{\partial x^2} - z = 0$ given $z = \cos y$, $\frac{\partial z}{\partial x} = \sin y$ when $x=0$.

Soln. Given $\frac{\partial^2 z}{\partial x^2} - z = 0$ — (1)

Assuming (1) as ODE

$$\frac{d^2 z}{dx^2} - z = 0$$

$$\Rightarrow (D^2 - 1)z = 0$$

$$\text{A.E } m^2 - 1 = 0$$

$$\Rightarrow m = \pm 1$$

$\therefore z = C_1 e^x + C_2 e^{-x}$ is solution of ODE.

The solution of PDE (1) is

$$z = f(y) e^x + g(y) e^{-x} \text{ — (2)}$$

diff (2) w.r.t 'x', treating 'y' constant

$$\frac{\partial z}{\partial x} = f(y) e^x - g(y) e^{-x} \text{ — (3)}$$

put $x=0$ in (2) & (3)

$$\left. \begin{array}{l} \text{(2)} \Rightarrow \cos y = f(y) + g(y) \\ \text{(3)} \Rightarrow \sin y = f(y) - g(y) \end{array} \right\} \text{ Solving, we get}$$

→

$$f(y) = \frac{\cos y + \sin y}{2}, \quad g(y) = \frac{\cos y - \sin y}{2}$$

$$\therefore \textcircled{1} \Rightarrow z = \frac{e^x}{2} (\cos y + \sin y) + \frac{e^{-x}}{2} (\cos y - \sin y)$$

$$z = \cos y \left(\frac{e^x + e^{-x}}{2} \right) + \sin y \left(\frac{e^x - e^{-x}}{2} \right)$$

$$\Rightarrow z = \underline{\underline{\cos y \cosh x + \sin y \sinh x}} \rightarrow \text{Required soln}$$

$$\textcircled{2} \text{ Solve } \frac{\partial^2 z}{\partial y^2} - 4 \frac{\partial z}{\partial y} + 4z = 0 \quad \text{given } z=0, \frac{\partial z}{\partial y} = e^x \text{ when } y=0.$$

soln: Given $\frac{\partial^2 z}{\partial y^2} - 4 \frac{\partial z}{\partial y} + 4z = 0$ — (1)

Assuming (1) as ODE

$$\frac{d^2 z}{dy^2} - 4 \frac{dz}{dy} + 4z = 0$$

$$\Rightarrow (D^2 - 4D + 4)z = 0$$

$$\text{A.E } m^2 - 4m + 4 = 0$$

$$\Rightarrow m = 2, 2$$

$$\therefore \text{Solution of ODE is } z = (c_1 + c_2 y) e^{2y}.$$

$$\text{The solution of PDE is } z = f(x) e^{2y} + y g(x) e^{2y} \text{ — (2)}$$

diff (2) wrt 'y' treating 'x' constant

$$\frac{\partial z}{\partial y} = 2f(x) e^{2y} + 2e^{2y} g(x) y + e^{2y} g(x) \text{ — (3)}$$

put $y=0$ in (2) & (3)

$$\textcircled{2} \Rightarrow \boxed{0 = f(x)}$$

$$\textcircled{3} \Rightarrow e^x = 2f(x) + e^0 g(x) \Rightarrow \boxed{g(x) = e^x}$$

substituting

$$(b) \Rightarrow z = y e^{x+y}$$

$$\Rightarrow \underline{z = y e^{x+y}} \rightarrow \text{Required soln}$$

(3) Solve $\frac{\partial^3 u}{\partial t^3} + 4 \frac{\partial u}{\partial t} = 0$ given $u=0, \frac{\partial u}{\partial t} = 0, \frac{\partial^2 u}{\partial t^2} = 4$
when $t=0,$

soln: given $\frac{\partial^3 u}{\partial t^3} + 4 \frac{\partial u}{\partial t} = 0$ — (1)

Assuming (1) as ODE

$$\frac{d^3 u}{dt^3} + 4 \frac{du}{dt} = 0$$

$$\Rightarrow (D^3 + 4D)u = 0$$

$$\text{A.E } m^3 + 4m = 0$$

$$\Rightarrow m = 0, \pm 2i$$

\therefore Solution of ODE is $u = C_1 + C_2 \cos 2t + C_3 \sin 2t$

\therefore The solution of PDE is $u = f(x) + g(x) \cos 2t + h(x) \sin 2t$ — (2)

diff (2) wrt 't', treating 'x' constant.

$$\frac{\partial u}{\partial t} = -2g(x) \sin 2t + 2h(x) \cos 2t$$
 — (3)

diff (3) wrt 't' for

$$\frac{\partial^2 u}{\partial t^2} = -4g(x) \cos 2t - 4h(x) \sin 2t$$
 — (4)

put $t=0$ in (2), (3) & (4)

$$(2) \Rightarrow 0 = f(x) + g(x)$$

$$(3) \Rightarrow 0 = 2h(x) \Rightarrow \boxed{h(x) = 0}$$

$$(4) \Rightarrow 4 = -4g(x) \Rightarrow \boxed{g(x) = -1} \Rightarrow \boxed{f(x) = 1}$$

$$\therefore \textcircled{2} \Rightarrow u = 1 - \cos \pi t$$

$$u = \underline{\underline{\pi \sin^2 t}} \text{ — Required soly}$$

$$\textcircled{4} \text{ solve } \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x}$$

soly given $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x}$

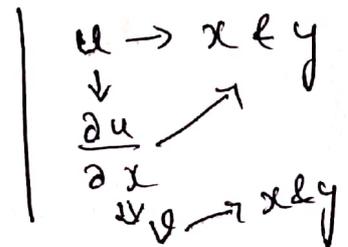
$$\Rightarrow \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial u}{\partial x} \text{ — } \textcircled{1}$$

\textcircled{1} is homogeneous PDE involving the derivative w.r.t x & y (both independent variables).

\therefore we need to go for substitution & reduce.

put $\frac{\partial u}{\partial x} = v$

$$\textcircled{1} \Rightarrow \frac{\partial v}{\partial y} = v \text{ — } \textcircled{2}$$



\textcircled{2} is homogeneous PDE involving derivative w.r.t one independent variable only.

Assuming \textcircled{2} as ODE

$$\frac{dv}{dy} - v = 0$$

$$\Rightarrow (D-1)v = 0$$

$$\text{A.E } m-1=0$$

$$\Rightarrow m=1$$

solution of ODE is $v = C_1 e^y$

\therefore The solution of PDE \textcircled{2} is $v = f(x) e^y$

$$\Rightarrow \frac{\partial u}{\partial x} = f(x) e^y \text{ — } \textcircled{3}$$

(3) is Non-homogeneous PDE.
On direct integration we get

$$z = F(x)e^y + g(y) \quad \text{--- (4)}$$

$$\text{where } F(x) = \int f(x) dx$$

\therefore (4) is the required soln of (1).

(5) Solve $\frac{\partial^3 u}{\partial x^2 \partial y} - \frac{\partial u}{\partial y} = 0$ given that $u = \frac{x^2}{2}$ when $y=0$,
 $\frac{\partial u}{\partial y} = 0$, $\frac{\partial^2 u}{\partial x \partial y} = y$ when $x=0$.

soln.

$$\text{Given } \frac{\partial^3 u}{\partial x^2 \partial y} - \frac{\partial u}{\partial y} = 0 \quad \text{--- (1)}$$

$$\text{put } \frac{\partial u}{\partial y} = v$$

$$\text{(1)} \Rightarrow \frac{\partial^2 v}{\partial x^2} - v = 0 \quad \text{--- (2)}$$

assuming (2) as ODE

$$\frac{d^2 v}{dx^2} - v = 0$$

$$\Rightarrow (D^2 - 1)v = 0$$

$$\text{A.E } m^2 - 1 = 0$$

$$\Rightarrow m = \pm 1$$

solution of ODE is $v = C_1 e^x + C_2 e^{-x}$

\therefore The solution of PDE (2) is $v = f(y)e^x + g(y)e^{-x}$

$$\Rightarrow \frac{\partial u}{\partial y} = f(y)e^x + g(y)e^{-x} \quad \text{--- (3)}$$

integrate (3) wrt 'y' treating 'x' constant

$$u = F(y)e^x + G(y)e^{-x} + h(x) \quad \text{--- (4)}$$

where $F(y) = \int f(y) dy$, $G(y) = \int g(y) dy$.

diff (3) wrt 'x' part

$$\frac{\partial u}{\partial x \partial y} = f(y)e^x - g(y)e^{-x} \quad \text{--- (5)}$$

put $x=0$ in (3) & (5)

$$\begin{aligned} (3) &\Rightarrow 0 = f(y) + g(y) \\ (5) &\Rightarrow y = f(y) - g(y) \end{aligned} \quad \left. \vphantom{\begin{aligned} (3) \\ (5) \end{aligned}} \right\} \text{solving}$$

$$f(y) = \frac{y}{2}$$

$$g(y) = -\frac{y}{2}$$

$$F(y) = \frac{y^2}{4}$$

$$G(y) = -\frac{y^2}{4}$$

put $y=0$ in (4)

$$(4) \Rightarrow \frac{x^2}{2} = 0 + 0 + h(x)$$

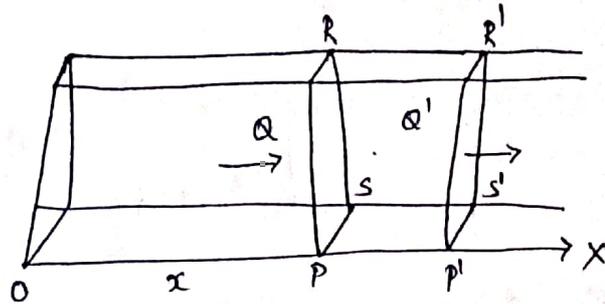
$$\Rightarrow h(x) = \frac{x^2}{2}$$

$$\therefore (4) \Rightarrow u = \frac{e^x y^2}{4} - \frac{e^{-x} y^2}{4} + \frac{x^2}{2}$$

$$u = \frac{y^2}{2} \left(\frac{e^x - e^{-x}}{2} \right) + \frac{x^2}{2}$$

$$\Rightarrow u = \frac{y^2 \sinh x + x^2}{2} \rightarrow \text{Required soln}$$

Derivation of one-dimensional heat equation:-



Consider a homogeneous bar of constant cross-sectional area A . Let ρ be the density, s be the specific heat and k be the thermal conductivity of the material. Let the sides be insulated so that the stream lines of heat flow are parallel and perpendicular to the area A . Let one end of the bar be taken as the origin O and the direction of the heat flow be the positive x -axis. Let $u = u(x, t)$ be the temperature of the slab at a distance x from the origin.

Consider an element of bar between the planes $PQRS$ and $P'Q'R'S'$ at a distance x and $x + \delta x$ from the end O .

Let δu be the change in temperature in a slab of thickness δx of the bar.

$$\text{Mass of the element} = A \rho \delta x$$

$$\text{Quantity of heat stored in this slab element} = A \rho s \delta x \delta u$$

Hence, the rate of increase of heat in this slab element

$$\text{is } R = (A \rho s \delta x) \frac{\partial u}{\partial t} \quad \text{--- (1)}$$

If R_1 is the rate of inflow of heat and R_0 is the rate of outflow of heat, we have

$$R_1 = -kA \left[\frac{\partial u}{\partial x} \right]_x \quad \text{and} \quad R_0 = -kA \left[\frac{\partial u}{\partial x} \right]_{x+\delta x} \quad \text{--- (2)}$$

where $-ve$ sign is due to flow of heat from higher

temperature to lower temperature.

From (1) & (2), $R = R_3 - R_0$

$$\rho \delta x \frac{\partial u}{\partial t} = kA \left[\frac{\partial u}{\partial x} \right]_{x+\delta x} - kA \left[\frac{\partial u}{\partial x} \right]_x$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{k}{\rho \delta} \left[\frac{\left(\frac{\partial u}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial u}{\partial x} \right)_x}{\delta x} \right]$$

taking limit as $\delta x \rightarrow 0$ in RHS

$$\frac{\partial u}{\partial t} = \frac{k}{\rho \delta} \lim_{\delta x \rightarrow 0} \left[\frac{\left(\frac{\partial u}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial u}{\partial x} \right)_x}{\delta x} \right]$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{k}{\rho \delta} \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right)$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

$$\text{where } c^2 = \frac{k}{\rho \delta}$$

is the required 1 dimensional heat equation.

Solution of 1-dimensional heat equation by the method of separation of variables:-

One dimensional heat equation is given by $u_t = c^2 u_{xx}$

$$\text{i.e. } \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{--- (1)}$$

let $u = XT$ where $X = X(x)$, $T = T(t)$ be the solution
L (2) of (1)

$$\text{put (2) in (1), (2) } \Rightarrow \frac{\partial}{\partial t}(XT) = c^2 \frac{\partial^2}{\partial x^2}(XT)$$

$$\Rightarrow X \frac{\partial T}{\partial t} = c^2 T \frac{\partial^2 X}{\partial x^2}$$

$$\div X c^2 T$$

$$\Rightarrow \frac{1}{c^2 T} \frac{\partial T}{\partial t} = \frac{1}{X} \frac{\partial^2 X}{\partial x^2} = K$$

$$\Rightarrow \frac{1}{c^2 T} \frac{\partial T}{\partial t} = K$$

$$\Rightarrow \frac{dT}{dt} - Kc^2 T = 0$$

$$\Rightarrow (D - Kc^2)T = 0 \text{ --- (3)}$$

where $D = \frac{d}{dt}$

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = K$$

$$\Rightarrow \frac{d^2 X}{dx^2} - KX = 0$$

$$\Rightarrow (D^2 - K)X = 0 \text{ --- (4)}$$

where $\frac{d^2}{dx^2} = D^2$

Case (i) :- when K is zero ($K=0$)

$$(3) \Rightarrow DT = 0$$

$$\text{A.E } m=0$$

$$T = C_1$$

$$(4) \Rightarrow D^2 X = 0$$

$$\text{A.E } m^2 = 0$$

$$\Rightarrow m = 0, 0$$

$$X = C_2 + C_3 X$$

$\therefore (2) \Rightarrow u = (C_2 + C_3 X) C_1$ is the solution of (1) when K is zero. (5)

Case (ii) :- when K is positive, $K = +p^2$

$$(3) \Rightarrow (D + p^2 c^2)T = 0$$

$$\text{A.E } m - p^2 c^2 = 0$$

$$\Rightarrow m = p^2 c^2$$

$$T = C_4 e^{p^2 c^2 t}$$

$$(4) \Rightarrow (D^2 - p^2)X = 0$$

$$\text{A.E } m^2 - p^2 = 0$$

$$\Rightarrow m = \pm p$$

$$X = C_5 e^{pX} + C_6 e^{-pX}$$

$\therefore (2) \Rightarrow u = (C_5 e^{pX} + C_6 e^{-pX}) (C_4 e^{p^2 c^2 t})$ is the soln of (1) when K is positive. (6)

Case (iii):- when k is $-ve$ $k = -p^2$

$$(3) \Rightarrow (D + p^2 c^2) T = 0$$

$$\text{A.E } m + p^2 c^2 = 0$$

$$\Rightarrow m = -p^2 c^2$$

$$T = c_7 e^{-p^2 c^2 t}$$

$$(4) \Rightarrow (D^2 + p^2) X = 0$$

$$\text{A.E } m^2 + p^2 = 0$$

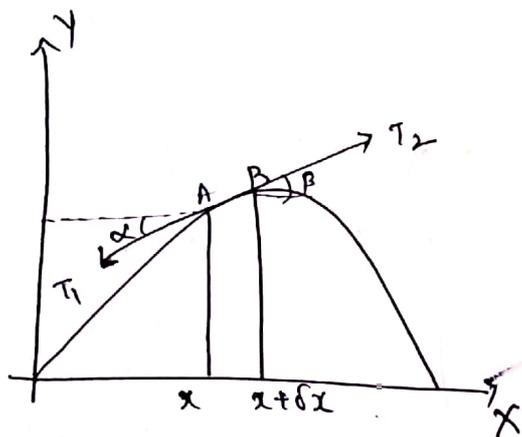
$$\Rightarrow m = \pm pi$$

$$X = c_8 \cos px + c_9 \sin px$$

\therefore (2) $\Rightarrow u = (c_8 \cos px + c_9 \sin px) (c_7 e^{-p^2 c^2 t})$ is the solution of (1) when k is negative. \hookrightarrow (7)

equations (5), (6) & (7) gives the various possible solutions of one dimensional heat equation.

Derivation of one dimensional wave equation:-



consider a flexible string tightly stretched between two fixed points of distance l apart. let ρ be the mass per unit length of the string.

Assumptions:-

- (1) The tension T of the string is same throughout.
- (2) The effect of gravity can be ignored due to large tension T
- (3) The motion of the string is in small transverse vibrations.

Consider the forces acting on a small element AB of length δx .
 Let T_1 and T_2 be the tensions at the points A and B.
 Since, there is no motion in the horizontal direction, the horizontal components T_1 and T_2 must cancel each other.

$$\therefore T_1 \cos \alpha = T_2 \cos \beta = T \quad \text{--- (1)}$$

where α and β are the angles made by T_1 and T_2 with the horizontal. Vertical components of tension are $-T_1 \sin \alpha$ and $T_2 \sin \beta$, where -ve sign is used because T_1 is directed downwards. Hence, the resultant force acting vertically upwards is $T_2 \sin \beta - T_1 \sin \alpha$.

By Newton's second law of motion,

Force = mass \times acceleration

$$T_2 \sin \beta - T_1 \sin \alpha = (\rho \delta x) \left(\frac{\partial^2 u}{\partial t^2} \right)$$

$\div T$

$$\frac{T_2}{T} \sin \beta - \frac{T_1}{T} \sin \alpha = \frac{\rho \delta x}{T} \left(\frac{\partial^2 u}{\partial t^2} \right)$$

from (1), $\frac{T_1}{T} = \cos \alpha$, $\frac{T_2}{T} = \cos \beta$

$$\therefore \frac{\sin \beta}{\cos \beta} - \frac{\sin \alpha}{\cos \alpha} = \frac{\rho \delta x}{T} \left(\frac{\partial^2 u}{\partial t^2} \right)$$

$$\Rightarrow \tan \beta - \tan \alpha = \frac{\rho \delta x}{T} \left(\frac{\partial^2 u}{\partial t^2} \right) \quad \text{--- (2)}$$

But $\tan \beta = \left(\frac{\partial u}{\partial x} \right)_{x+\delta x}$ — slope at B ($x+\delta x$)

$\tan \alpha = \left(\frac{\partial u}{\partial x} \right)_x$ — slope at A (x)

$$\text{(2)} \Rightarrow \left(\frac{\partial u}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial u}{\partial x} \right)_x = \frac{\rho \delta x}{T} \left(\frac{\partial^2 u}{\partial t^2} \right)$$

∴ δx & taking limit as $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} \right] = \frac{f}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{f}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = \frac{T}{f} \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

is the required one dimensional wave equation.

$$\text{where } c^2 = \frac{T}{f}$$

Solution of one-dimensional wave equation by the method of separation of variables:-

One dimensional wave equation is given by $u_{tt} = c^2 u_{xx}$

$$\text{i.e. } \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{--- (1)}$$

Let $u = XT$ where $X = X(x)$, $T = T(t)$ be the solution of (1)

(2) in (1)

$$(1) \Rightarrow \frac{\partial^2}{\partial t^2} (XT) = c^2 \frac{\partial^2}{\partial x^2} (XT)$$

$$\Rightarrow X \frac{\partial^2 T}{\partial t^2} = c^2 T \frac{\partial^2 X}{\partial x^2}$$

$$\div X c^2 T$$

$$\Rightarrow \frac{1}{c^2 T} \frac{\partial^2 T}{\partial t^2} = \frac{1}{X} \frac{\partial^2 X}{\partial x^2} = k$$

$$\frac{1}{c^2 T} \frac{\partial^2 T}{\partial t^2} = k$$

$$\Rightarrow \frac{d^2 T}{dt^2} - k c^2 T = 0$$

$$\Rightarrow (D^2 - k c^2) T = 0 \quad \text{--- (3)}$$

where $D = \frac{d}{dt}$

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = k$$

$$\frac{d^2 X}{dx^2} - k X = 0$$

$$(D^2 - k) X = 0 \quad \text{--- (4)}$$

where $D = \frac{d}{dx}$

Case (i):- when $k = 0$

$$(3) \Rightarrow D^2 T = 0$$

$$\text{A.E } m^2 = 0$$

$$\Rightarrow m = 0, 0$$

$$T = C_1 + C_2 t$$

$$(4) \Rightarrow D^2 X = 0$$

$$\text{A.E } m^2 = 0$$

$$\Rightarrow m = 0, 0$$

$$X = C_3 + C_4 x$$

(2) $\Rightarrow u = (C_3 + C_4 x) (C_1 + C_2 t)$ is the solution of (1) when $k = 0$ \leftarrow (5)

Case (ii):- when k is positive ($k = p^2$)

$$(3) \Rightarrow (D^2 - p^2 c^2) T = 0$$

$$\text{A.E } m^2 - p^2 c^2 = 0$$

$$\Rightarrow m = \pm pc$$

$$T = C_5 e^{pct} + C_6 e^{-pct}$$

$$(4) \Rightarrow (D^2 - p^2) X = 0$$

$$\text{A.E } m^2 - p^2 = 0$$

$$\Rightarrow m = \pm p$$

$$X = C_7 e^{px} + C_8 e^{-px}$$

(2) $\Rightarrow u = (C_7 e^{px} + C_8 e^{-px}) (C_5 e^{pct} + C_6 e^{-pct})$ is the solution of (1) when k is positive. \leftarrow (6)

case (iii): when k is negative ($k = -p^2$)

$$(3) \Rightarrow (D^2 + p^2 c^2) T = 0$$

$$\text{A.E } m^2 + p^2 c^2 = 0$$

$$\Rightarrow m = \pm pci$$

$$T = C_9 e^{\cos p c t} + C_{10} \sin p c t$$

$$(4) \Rightarrow (D^2 + p^2) X = 0$$

$$\text{A.E } m^2 + p^2 = 0$$

$$\Rightarrow m = \pm pi$$

$$X = C_{11} \cos p x + C_{12} \sin p x$$

(5) $\Rightarrow u = (C_{11} \cos p x + C_{12} \sin p x) (C_9 \cos p c t + C_{10} \sin p c t)$ is the solution of (1) when k is negative. \hookrightarrow (7)

Equations (5), (6) & (7) give the possible solutions of one dimensional wave equation.

X

X