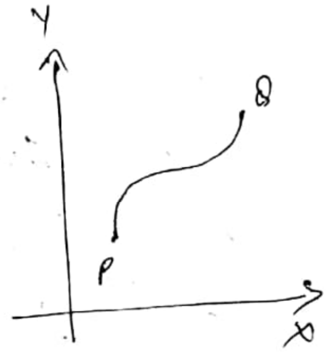
Complex Integration. (1)

* The complex line integral along the path 'C' equally denoted by $\int_C f(z)dz$.
* If 'C' is a simple closed curve the notation $\oint_C f(z)dz$ is also used.

Properties of complex integral



* If '-C' denotes the curve traversed from Q to P then

$$\int_{-C} f(z)dz = -\int_C f(z)dz$$

* If C is split into a no of parts C_1, C_2, C_3 --- then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + ----$$

* If λ_1 and λ_2 are constants then

$$\int_C [\lambda_1 f_1(z) \pm \lambda_2 f_2(z)]\, dz = \lambda_1 \int_C f_1(z)dz \pm \lambda_2 \int_C f_2(z)dz$$

Line integral of a complex valued function.

Let $f(z) = u(x,y) + iv(x,y)$ be a complex valued fun defined over a region R and C be a curve in the region. Then

$$\int_C f(z)dz = \int_C (u+iv)(dx+idy)$$

ie $$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx + udy)$$

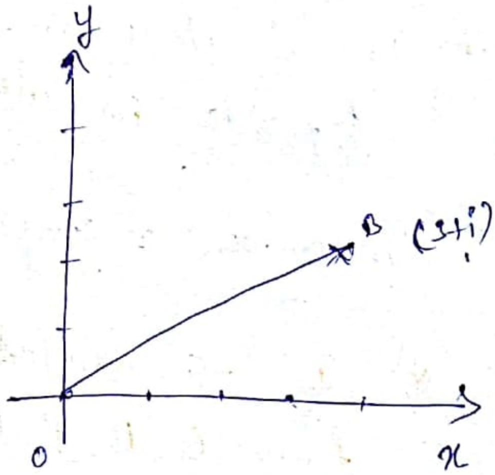
This shows that the evaluation of a line integral of a complex valued fun is nothing but the evaluation of line integrals of real valued functions.

1) Evaluate $\int_C z^2 dz$

a) along the straight line from $z=0$ to $z=3+i$

b) along the curve made up of two line segments, one from $z=0$ to $z=3$ and another from $z=3$ to $z=3+i$.

Sol^n a) $\int_C z^2 dz = \int_{z=0}^{3+i} z^2 dz$



Here z varies from 0 to 3+i means that (x, y) varies from (0, 0) to (3, 1). The eq^n of the line joining (0, 0) and (3, 1) is given by

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

$$\frac{y-0}{x-0} = \frac{1-0}{3-0} \text{ (or) } y = \frac{x}{3} \text{ (or) } \underline{x = 3y}$$

further $z^2 = (x+iy)^2 = x^2 + i^2y^2 + 2ixy = x^2 - y^2 + i(2xy)$

and $dz = dx + i\,dy$

$$\int_C z^2 dz = \int_{(0,0)}^{(3,1)} \{(x^2-y^2) + i(2xy)\}\{dx + i\,dy\}$$

$$= \int_{(0,0)}^{(3,1)} (x^2-y^2)dx - 2xy\,dy + i\int_{(0,0)}^{(3,1)} \{2xy\,dx + (x^2-y^2)dy\}$$

we have $y = \frac{x}{3}$ (or) $x = 3y$ and we shall convert these integrals into the variable i and integrate w.r.t y from 0 to 1. we also have $\underline{dx = 3dy}$

$$\therefore \int_C z^2 dz = \int_{y=0}^{1} \{(9y^2 - y^2)\, 3dy - 2(3y)y\, dy\}$$

$$+ i \int_{y=0}^{1} \{2(3y)y\, 3dy + (9y^2 - y^2)\, dy\}$$

$$= \int_{y=0}^{1} (24y^2 - 6y^2)\, dy + i \int_{y=0}^{1} (18y^2 + 8y^2)\, dy$$

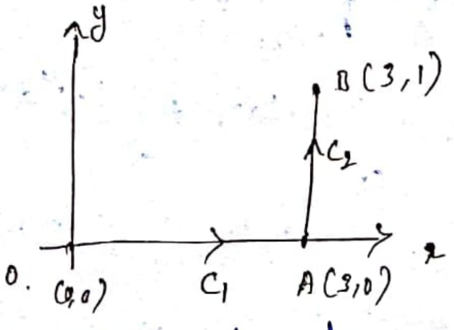
$$= \int_0^1 18y^2 dy + i \int_0^1 26y^2 dy$$

$$= 18\left[\frac{y^3}{3}\right]_0^1 + 26i\left[\frac{y^3}{3}\right]_0^1$$

$$= 6 + \frac{26}{3} i$$

Thus $\int_C z^2 dz = 6 + \frac{26i}{3}$ along the given path.

b) Segments from $z=0$ to $z=3$ and then from $z=3$ to $3+i$ means that (x, y) varies from $(0,0)$ to $(3,0)$ and then from $(3,0)$ to $(3,1)$ as shown in the fig.



$$\int_C z^2 dz = \int_{C_1} z^2 dz + \int_{C_2} z^2 dz \quad \text{——} (1)$$

Now along C_1 : $y = 0 \Rightarrow dy = 0$ and

$x \rightarrow 0$ to 3, $z^2 dz \longrightarrow x^2 dx$

Also along C_2 : $x = 3 \Rightarrow dx = 0$ and $y \longrightarrow 0$ to 1

$z^2 dz \longrightarrow (3+iy)^2\, i\, dy$ $\therefore$

$(1) \Rightarrow \int_C z^2 dz = \int_{x=0}^{3} x^2 dx + i\int_{y=0}^{1} (3+iy)^2 dy$

$= \frac{x^3}{3}\Big|_0^3 + i\int_{y=0}^{1} (9-y^2+6iy)\,dy$

$= 9 + i\left[9y - \frac{y^3}{3} + 3iy^2\right]_0^1$

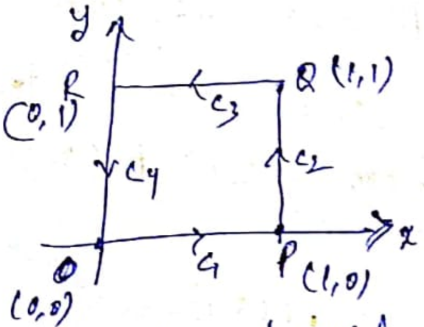
$= 9 + i\left(9 - \frac{1}{3} + 3i\right)$

$= (9-3) + i\cdot\frac{26}{3}$

Thus $\int_C z^2 dz = 6 + \frac{26}{3}i$ along the given path

2) Evaluate $\int_C |z|^2 dz$ where C is a square with following vertices $(0,0)\,(1,0),\,(1,1)\,(0,1)$.

>> The curve C is as shown in the following fig.



$\int_C |z|^2 dz = \int_{C_1} |z|^2 dz + \int_{C_2} |z|^2 dz + \int_{C_3} |z|^2 dz + \int_{C_4} |z|^2 dz \quad \text{--- (1)}$

we have $|z|^2 dz = (x^2+y^2)\,(dx+idy)$

Along OP C_1, $y=0 \Rightarrow dy=0$, $|z^2|dz = x^2 dx$ where $0 \le x \le 1$

Along PQ C_2, $x=1 \Rightarrow dx=0$, $|z^2|dz = (1+y^2)\,i\,dy$ where $0 \le y \le 1$

Along QR C_3, $y=1 \Rightarrow dy=0$, $|z^2|dz = (x^2+1)\,dx$ where $1 \le x \le 0$

Along RO C_4, $x=0 \Rightarrow dx=0$, $|z^2|\,dz = y^2(i\,dy)$ where $1 \le y \le 0$

using these (1) $\Rightarrow$

$$\int_C |z|^2 dz = \int_{x=0}^{1} x^2\,dx + i\int_{y=0}^{1} (1+y^2)\,dy + \int_{x=1}^{0} (x^2+1)\,dx + i\int_{y=1}^{0} y^2\,dy$$

$$= \frac{x^3}{3}\Big|_0^1 + i\left[y+\frac{y^3}{3}\right]_0^1 + \left[\frac{x^3}{3}+x\right]_1^0 + i\left[\frac{y^3}{3}\right]_1^0$$

$$= \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3}$$

$$= \underline{\underline{-1+i}}$$

Thus $\int_C |z|^2 dz = -1+i$ along the given path.

3) Evaluate $\int_0^{2+i} (\bar{z})^2\,dz$ along:

a) the line $x = 2y$

b) the real axis upto 2 and then vertically to $2+i$.

>> Let $I = \int_0^{2+i} (\bar{z})^2 dz$

we have $(\bar{z})^2 = (x-iy)^2 = (x^2-y^2) - i(2xy)$ —— (1)

and $dz = dx + i\,dy$ —— (2)

a) Along $x = 2y$, $dx = 2dy$

$z=0$ to $2+i$ $\Rightarrow$ (x,y) varies from $(0,0)$ to $(2,1)$ where $0 \le y \le 1$

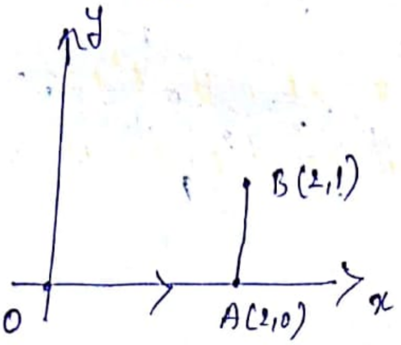
$$\therefore I = \int_{y=0}^{1} \left[(4y^2-y^2) - i\,4y^2\right](2dy + i\,dy)$$

$$= \int_0^1 (3-4i)\,y^2\,(2+i)\,dy$$

$$= \int_0^1 (10-5i)\,y^2\,dy = 5(2-i)\,\frac{y^3}{3}\Big|_0^1 = \frac{5}{3}(2-i)$$

Thus $I = \underline{\frac{5}{3}(2-i)}$ along the given path.

b) $I = \int_{OA} (\bar{z})^2 dz + \int_{AB} (\bar{z})^2 dz$ ---- (3)



Along OA where $O = (0,0)$ and $A = (2,0)$

$y = 0 \Rightarrow dy = 0$ and $0 \le x \le 2$

Along AB where $A = (2,0)$ and $B = (2,1)$

$x = 2 \Rightarrow dx = 0$ and $0 \le y \le 1$

From ① and ② we have

along OA, $(\bar{z})^2 dz = x^2 dx$; $0 \le x \le 2$

along AB. $(\bar{z})^2 dz = [(4-y^2) - 4iy]\, i\,dy$; $0 \le y \le 1$

$$\int_{OA} (\bar{z})^2 dz = \int_{x=0}^{2} x^2 dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3} \quad \text{--- (4)}$$

$$\int_{AB} (\bar{z})^2 dz = i\int_{y=0}^{1} [(4-y^2) - 4iy]\,dy$$

$$= i\left[4y - \frac{y^3}{3}\right]_0^1 + 4\left[\frac{y^2}{2}\right]_0^1$$

$$= 2 + \frac{11}{3}i \quad \text{—— (5)}$$

using ④ & ⑤

eqn (3) $\Rightarrow I = \frac{8}{3} + \left(2 + \frac{11}{3}i\right)$

Thus $I = \frac{1}{3}(14 + 11i)$ along the given path

4) Evaluate $\int_{(0,3)}^{(2,4)} (2y+x^2)\,dx + (3x-y)\,dy$ along the following path.

a) the parabola $x = 2t$, $y = t^2+3$

b) the st line from $(0,3)$ to $(2,4)$

>> a) x varies from 0 to 2 and hence

$$\left.\begin{array}{l} \text{if } x=0,\ 2t=0 \quad \therefore t=0 \\ \text{if } x=2,\ 2t=2 \quad \therefore t=1 \end{array}\right\} \Rightarrow t \to 0 \text{ to } 1$$

$$I = \int_{(0,3)}^{(2,4)} (2y+x^2)\,dx + (3x-y)\,dy$$

$$= \int_{t=0}^{1} \{2(t^2+3)+4t^2\}\,2dt + \{3(2t)-(t^2+3)\}\,2t\,dt$$

$$= \int_0^1 [2(6t^2+6) + (6t-t^2-3)2t]\,dt$$

$$= \int_0^1 (24t^2 - 2t^3 - 6t + 12)\,dt$$

$$= 24\left.\frac{t^3}{3}\right|_0^1 - 2\left.\frac{t^4}{4}\right|_0^1 - 6\left.\frac{t^2}{2}\right|_0^1 + 12t\Big|_0^1$$

$$= 8 - \frac{1}{2} - 3 + 12$$

$$= \frac{33}{2}$$

Thus $I = \frac{33}{2}$ along the given path.

b) Eqn of the st line joining (0,3) and (2,4) is given by $\frac{y-3}{x-0} = \frac{4-3}{2-0}$

ie $\frac{y-3}{x} = \frac{1}{2}$ (or) $x = 2y-6$ hence $dx = 2dy$

Now $I = \int_{y=3}^{4} \{2y + (2y-6)^2\} 2dy + \{3(2y-6) - y\} dy$

$= \int_{3}^{4} \{(4y^2 - 22y + 36)2 + (5y-18)\} dy$

$= \int_{3}^{4} (8y^2 - 39y + 54) dy$

$= \frac{97}{6}$

Thus $I = 97/6$ along the given path

5) Evaluate $\int_c \bar{z}\, dz$ where c represents the foll^ng paths

a) the straight line from $-i$ to i

b) the right half of the unit circle $|z| = 1$ from $-i$ to i

>> a) $z = x+iy$. $\therefore \bar{z} = x - iy$, $dz = dx + idy$

c is the st line joining the points $(0,-1)$ and $(0,1)$

here $x = 0 \Rightarrow dx = 0$, $y \to -1$ to $+1$.

$\int_c \bar{z} dz = \int_{y=-1}^{1} (x-iy)(dx+idy)$

$= \int_{-1}^{1} (-iy)\, i dy = \int_{-1}^{1} y dy = \frac{y^2}{2}\Big]_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$

Thus $\int \bar{z} dz = 0$ along the given path

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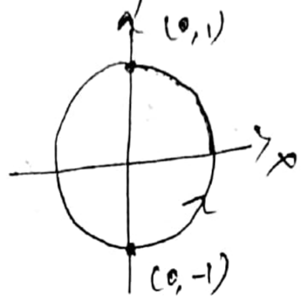
ii b) The curve C is shown in the foll^{ng} fig.

$C : |z| = 1$. we can take $z = e^{i\theta}$

Also $\bar{z} = e^{-i\theta}$ and $dz = ie^{i\theta}\,d\theta$

from the fig. $y \to -1$ to 1 and $x = 0$



But $x = \cos\theta$, $y = \sin\theta$

$y = -1$ $\sin\theta = -1$ $\therefore \theta = -\pi/2$

$y = +1$. $\sin\theta = 1$ $\therefore \theta = \pi/2$

Now $\displaystyle\int_C \bar{z}\,dz = \int_{\theta=-\pi/2}^{\pi/2} e^{-i\theta} . ie^{i\theta} d\theta$

$\displaystyle = i\int_{-\pi/2}^{\pi/2} 1 . d\theta = i\left[\theta\right]_{-\pi/2}^{\pi/2} = \pi i$

Thus $\displaystyle\int_C \bar{z}\,dz = \pi i$ along the given path.

6) if C is a $\odot$le with centre 'a' and radius 'r' then S.T

a) $\displaystyle\int_C \frac{dz}{z-a} = 2\pi i$ b) $\displaystyle\int_C (z-a)^n dz = 0$ if $n \neq -1$

(or)

Show that $\displaystyle\int_C (z-a)^n dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}$

where C is the $\odot$le $|z-a| = r$.

>> on the given circle $|z-a| = r$,

we have $z - a = re^{i\theta}$

hence $dz = ire^{i\theta}\,d\theta$

also $0 \leq \theta \leq 2\pi$

a) $\int_C \frac{dz}{z-a} = \int_{\theta=0}^{2\pi} \frac{ire^{i\theta}d\theta}{re^{i\theta}} = i\int_{\theta=0}^{2\pi} d\theta = i\,\theta\Big|_0^{2\pi} = 2\pi i$

Thus $\int_C \frac{dz}{z-a} = 2\pi i$

b) Also $\int_C (z-a)^n dz = \int_{\theta=0}^{2\pi} (re^{i\theta})^n\, ire^{i\theta}d\theta$

$= ir^{n+1}\int_{\theta=0}^{2\pi} e^{i(n+1)\theta}d\theta$

$= \cancel{i}r^{n+1}\left[\frac{e^{i(n+1)\theta}}{\cancel{i}(n+1)}\right]_0^{2\pi}$

$= \frac{r^{n+1}}{n+1}\left[e^{i(n+1)2\pi} - 1\right]$

But $e^{i(n+1)2\pi} = \cos(n+1)2\pi + i\sin(n+1)2\pi$

$= 1 + i.0 = 1$

$\because \cos 2k\pi = 1$ and $\sin 2k\pi = 0$ for $k = 1, 2, 3 \ldots$

hence $\int_C (z-a)^n dz = \frac{r^{n+1}}{n+1}[1-1] = 0$ when $n \neq -1$

Thus we have proved that,

$$\int_C (z-a)^n dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if } n \neq -1. \end{cases}$$

Cauchy's theorem

Stmt:- If $f(z)$ is analytic at all points inside and on a simple closed curve C then $\int_C f(z)dz = 0$.

Proof: Let $f(z) = u+iv$

then $\int_C f(z)dz = \int_C (u+iv)(dx+idy)$

ie $\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx + udy)$ —— (1)

we have Green's theorem in a plane stating that if $M(x,y)$ and $N(x,y)$ are two real valued funs having Continuous first order p. derivatives in a region R bounded by the curve C then

$$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Applying this theorem to the two line integrals in the RHS of (1) we obtain

$$\int_C f(z)dz = \iint_R \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy$$

Since $f(z)$ is analytic, we have C-R eqns.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ and hence we have

$$\int_C f(z)dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dxdy$$

Thus we get $\int_C f(z)dz = 0$

This proves Cauchy's theorem.

consequences of Cauchy's theorem

* Stmt$_1$:- If $f(z)$ is analytic in a region R and if P and Q are any two points in it then $\int_P^Q f(z)\,dz$ is independent of the path joining P and Q. That is $\int_P^Q f(z)\,dz$ is same for all curves joining P and Q.

* Stmt$_2$:- If C_1, C_2 are two simple closed curves such that C_2 lies entirely within C_1 and if $f(z)$ is analytic on C_1, C_2 and in the region bounded by C_1, C_2 then
$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

* Stmt$_3$:- If C is a simple closed curve enclosing non overlapping simple closed curves $C_1, C_2, C_3 \ldots C_n$ and if $f(z)$ is analytic in the annular region b/w C and these curves then
$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$$

working procedure

* we need to evaluate the integrals of the form $\int_C \frac{f(z)}{z-a}\,dz$; $\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$ over a given closed curve C.
* Firstly we have to find out whether the point $z=a$ lies inside (or) outside the given curve C.
* If $z=a$ is inside C then we use Cauchy's integral formula in the forms $\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$ and $\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$
* If the point $z=a$ is outside C we can conclude that $\int_C f(z)\,dz = 0$ by Cauchy theorem.

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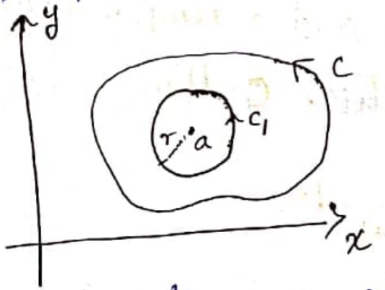
Cauchy's integral formula.

If $f(z)$ is analytic inside and on a simple closed curve C and if 'a' is any point within C then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

Proof: Since 'a' is a point within C, we shall enclose it by a circle C_1 with $z=a$ as centre and r as radius such that C_1 lies entirely within C,

The fun $\frac{f(z)}{z-a}$ is analytic inside and on the boundary of the annular region b/w C and C_1.



Now, as a consequence of cauchy's theorem,

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz \quad \text{----} (1)$$

The eqn of C_1 (circle with centre 'a' and radius r) can be written in the form $|z-a| = r$. That is

$$z-a = re^{i\theta} \quad (\text{or}) \quad z = a+re^{i\theta}$$

$$0 \le \theta \le 2\pi$$

$$dz = ire^{i\theta}d\theta$$

$\therefore (1) \Rightarrow$

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\,d\theta$$

ie $$\int_C \frac{f(z)}{z-a}\,dz = i\int_{\theta=0}^{2\pi} f(a+re^{i\theta})\,d\theta$$

This is true for any $r>0$ however small, hence as $r \to 0$ we get.

$$\int_C \frac{f(z)}{z-a}\,dz = i\int_{\theta=0}^{2\pi} f(a)\,d\theta = i f(a)\,\theta\Big|_0^{2\pi} = 2\pi i f(a)$$

Thus $$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$ [Cauchy's integral formula]

Generalized Cauchy's integral formula.

If $f(z)$ is analytic inside and a simple closed curve C and if 'a' is a point within C then

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Proof: we have cauchy's integral formula.

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}\,dz \quad \text{---- (1)}$$

Applying Leibnitz rule for diff. under the integral sign we have

$$f'(a) = \frac{1}{2\pi i}\int_C f(z)\cdot\frac{\partial}{\partial a}\left[\frac{1}{z-a}\right]dz$$

$$f'(a) = \frac{1}{2\pi i}\int_C f(z)\cdot\{(-1)\cdot(z-a)^{-2}\cdot(-1)\}\,dz$$

$$f'(a) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2} \quad \text{---- (2)}$$

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∴ Applying L. rule again for (2) we obtain

$$f''(a) = \frac{1!}{2\pi i}\int_C f(z)\cdot\frac{\partial}{\partial a}\left[(z-a)^{-2}\right]dz$$

$$= \frac{1!}{2\pi i}\int_C f(z)\cdot(-2)(z-a)^{-3}(-1)\,dz$$

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz$$

Continuing like this, after diff n times, we get

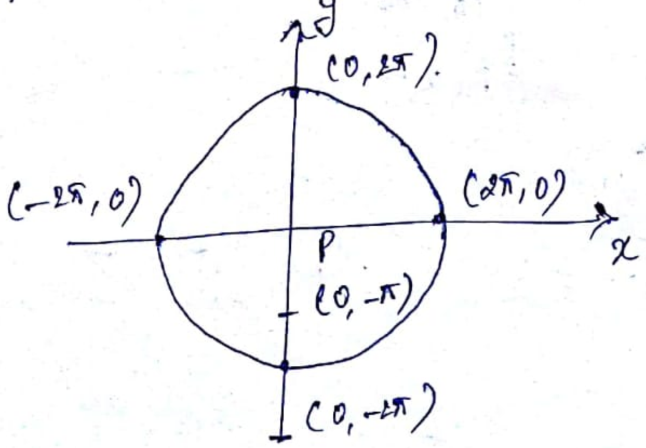
$$\boxed{f^{n}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz}$$

Here $f^{(n)}(a)$ denotes the nth derivative of $f(z)$ at $z=a$.

1) Evaluate $\int_C \frac{e^z}{z+i\pi}\,dz$ over each of the foll^ng contours C: a) $|z|=2\pi$ b) $|z|=\pi/2$ c) $|z-1|=1$

→ we have to evaluate the integral which can be written in the form $\int_C \frac{e^z}{z-(-i\pi)}\,dz$ which is of the form $\int_C \frac{f(z)}{z-a}\,dz$

here $f(z)=e^z$, $a=-i\pi$



a) $|z| = 2\pi$ is a circle with centre origin and radius 2π.

The point $z = a = -i\pi$ is the point $(0, -\pi)$ lies within the circle $|z| = 2\pi$

we have cauchy's integral formula $\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$

we have $f(z) = e^z$, $a = -i\pi$

$$\therefore \int_C \frac{e^z}{z+i\pi} dz = 2\pi i f(-i\pi) = 2\pi i e^{-i\pi} = 2\pi i(\cos\pi - i\sin\pi)$$
$$= -2\pi i$$

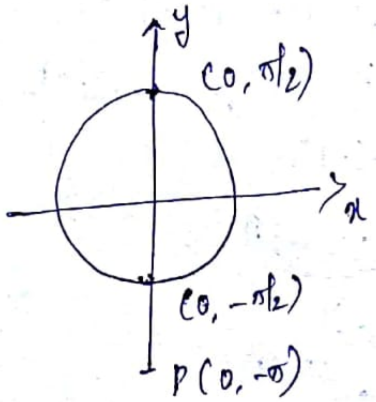
Thus $\int_C \frac{e^z}{z+i\pi} dz = -2\pi i$,

where C is the circle $|z| = 2\pi$.

b) $|z| = \pi/2$ is a circle with centre origin and radius $\pi/2$,

The point $P(0, -\pi)$ lies outside the circle $|z| = \pi/2$

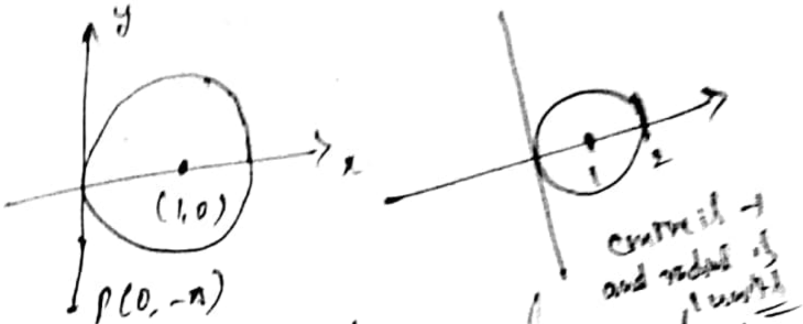
and $\frac{e^z}{z+i\pi}$ is analytic inside and on the circle $|z| = \pi/2$.



By Cauchy's theorem

$\int_C \frac{e^z}{z+i\pi} dz = 0$, where $C: |z| = \pi/2$

c) $|z-1| = 1$ is a cle. with centre at $z = a = 1$ and radius 1. That is a cle. with centre $(1, 0)$ and radius 1. ⑨



$|z-1|=1$

The point $p(0, -\pi)$ lies outside the cle $|z-1| = 1$ and hence by Cauchy's theorem

$$\int_C \frac{e^z}{z+i\pi}\, dz = 0, \quad \text{where } |z-1| = 1.$$

2) Evaluate $\int_C \frac{dz}{z^2-4}$ over the following curves C.

a) $C : |z| = 1$ b) $C : |z| = 3$ c) $C : |z+2| = 1$

>> Consider $\frac{1}{z^2-4} = \frac{1}{(z^2-2^2)} = \frac{1}{(z+2)(z-2)}$

Resolving into partial fractions,

$$\frac{1}{(z-2)(z+2)} = \frac{A}{(z-2)} + \frac{B}{(z+2)}$$

(or) $1 = A(z+2) + B(z-2)$

putting $z = 2$: $1 = A(4)$ $\therefore A = 1/4$

$z = -2$: $1 = B(-4)$ $\therefore B = -1/4$

Now $\frac{1}{(z-2)(z+2)} = \frac{1}{4}\cdot\frac{1}{z-2} - \frac{1}{4}\,\frac{1}{z+2}$

$$\therefore \int_C \frac{dz}{(z-2)(z+2)} = \frac{1}{4}\int_C \frac{dz}{z-2} - \frac{1}{4}\int_C \frac{dz}{z-(-2)} \quad \text{—— (1)}$$

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a) $c: |z| = 1$;

$\Rightarrow z = a = 2$. and $z = a = -2$ bothe of them lie outside c

Thus by Cauchy's theorem $\int_C \frac{dz}{z^2-4} = 0$ whee $c: |z| = 1$

b) $c: |z| = 3$; $z = a = 2$ and $z = a = -2$ lies inside the ⊙le, Also in each of the integrals as in the RHS of ①,

$f(z) = 1$

Aplying Cauchy's integral formula

$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$ we obtain

$\int_C \frac{dz}{z-2} = 2\pi i f(2) = 2\pi i \cdot (1) = 2\pi i$

$\int_C \frac{dz}{z+2} = 2\pi i f(-2) = 2\pi i \cdot (1) = 2\pi i$

Substituting these in the RHS. of ① we have

$\int_C \frac{dz}{z^2-4} = \frac{1}{4}(2\pi i) - \frac{1}{4}(2\pi i) = 0$

Thus $\int_C \frac{dz}{z^2-4} = 0$ where $c: |z| = 3$

c) $c: |z+2| = 1$. This is a ⊙le with centre $(-2, 0)$ and radius 1.

Let $A = (-2, 0)$ and $P = (2, 0)$ hence $AP = \sqrt{4} = 2 > 1$

$\therefore$ the point $z = a = 2$ lies outside the ⊙le and clearly the point $z = a = -2$ being $(-2, 0)$ lies inside the ⊙le.

hence by Cauchy's theorem $\int_C \frac{dz}{z-2} = 0$

$\therefore$ Also by Cauchy's integral formula,

$$\int_C \frac{dz}{z+2} = \int_C \frac{dz}{z-(-2)} = 2\pi i\, f(-2) \text{ where } f(z) = 1$$

$$\therefore \int_C \frac{dz}{z+2} = 2\pi i, 1 = 2\pi i$$

Substituting these value in the RHS of (1) we have,

$$\int_C \frac{dz}{z^2-4} = \frac{1}{4}\cdot 0 - \frac{1}{4}\cdot 2\pi i = \frac{-\pi i}{2}$$

Thus $\int_C \frac{dz}{z^2-4} = \frac{-\pi i}{2}$ where $C: |z+2| = 1$

3) Evaluate $\int_C \frac{e^z}{z-i\pi}$ where C is the circle a) $|z| = 2\pi$ b) $|z| = \pi/2$

a) $\int_C \frac{e^z}{z-i\pi} dz = -2\pi i$ for $C: |z| = 2\pi$

b) $\int_C \frac{e^z}{z-i\pi} dz = 0$ for $C: |z| = \pi/2$

Similar to problem (1)

4) Evaluate $\int_C \frac{e^{2z}}{(z+1)(z-2)} dz$ where C is the circle $|z| = 3$

>> The points $z = a = -1$, $z = a = 2$ being $(-1,0)$ $(2,0)$ lies inside $|z| = 3$

Now we shall resolve $\frac{1}{(z+1)(z-2)}$ into p. fractions.

Let $\frac{1}{(z+1)(z-2)} = \frac{A}{(z+1)} + \frac{B}{(z-2)}$

(or) $1 = A(z-2) + B(z+1)$

put $z = 2$, $B = 1/3$

$z = -1$, $A = -1/3$

$$\therefore \int_c \frac{e^{2z}\,dz}{(z+1)(z-2)} = \frac{1}{3}\left[\int_c \frac{e^{2z}}{z-2}\,dz - \int_c \frac{e^{2z}}{z+1}\,dz\right] \quad \text{—} (1)$$

we have Cauchy's integral formula.

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

taking $f(z) = e^{2z}$ and $a = 2, -1$ respectively we obtain

$$\int_c \frac{e^{2z}}{z-2}\,dz = 2\pi i\, f(2) = 2\pi i e^4$$

and $$\int_c \frac{e^{2z}}{z+1}\,dz = 2\pi i\, f(-1) = 2\pi i e^{-2} = \frac{2\pi i}{e^2}$$

substituting these in the RHS of (1) we obtain

$$\int_c \frac{e^{2z}\,dz}{(z+1)(z-2)} = \frac{1}{3}\left[2\pi i e^4 - \frac{2\pi i}{e^2}\right]$$

Thus $$\int_c \frac{e^{2z}\,dz}{(z+1)(z-2)} = \frac{2\pi i}{3}\left[e^4 - \frac{1}{e^2}\right]$$

5) Evaluate $\int_c \frac{e^{3z}}{z^2}\,dz$ over $c : |z| = 1$

>> The point $z = 0$ lies within the circle $|z| = 1$ and we have Cauchy's integral formula in the generalised form.

$$\int_c \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(a)$$

taking $f(z) = e^{3z}$, $a = 0$, $n = 1$ in this formula we obtain

$$\int_c \frac{e^{3z}}{z^2}\,dz = \frac{2\pi i}{1!} f'(a) \quad ; \quad \text{also } f'(z) = 3e^{3z}$$

$$\therefore \int_c \frac{e^{3z}}{z^2}\,dz = 2\pi i\,(3e^0) = 2\pi i(3) = 6\pi i$$

Thus $$\int_c \frac{e^{3z}}{z^2}\,dz = 6\pi i$$

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6). Evaluate $\int_c \frac{z^2+z+1}{(z-2)^3} dz$ over $C : |z| = 3$

>> The point $z = 2$ lies inside the circle $|z| = 3$

we have generalised Cauchy's integral formula.

$$\int_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$$

taking $f(z) = z^2+z+1$, we obtain $f''(z) = 2$.

$\therefore f''(2) = 2$ also by taking $a = 2$, $n = 2$ we have

$$\int_c \frac{z^2+z+1}{(z-2)^3} dz = \frac{2\pi i}{2!} f''(2) = \frac{2\pi i}{2} \cdot 2 = 2\pi i$$

Thus $\int_c \frac{z^2+z+1}{(z-2)^3} dz = 2\pi i$

7) Evaluate $\int_c \frac{e^{\pi z}}{(2z-i)^3} dz$ where C is the circle $|z| = 1$

>> we can write the given integral in the form

$$\int_c \frac{e^{\pi z}}{[2(z-i/2)]^3} = \frac{1}{8} \int_c \frac{e^{\pi z}}{(z-i/2)^3}$$

The point $z = i/2$ being $(0, 1/2)$ lies within the circle $|z| = 1$. we have generalised Cauchy's integral formula

$$\int_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$$

taking $f(z) = e^{\pi z}$, $a = i/2$, $n = 2$ we have

$$\int_c \frac{e^{\pi z}}{(z-i/2)^3} dz = \frac{2\pi i}{2!} f''(i/2) = \pi i f''(2)$$

x'ly by 1/8 we have

$$\frac{1}{8} \int_c \frac{e^{\pi z}}{(z-i/2)^3} dz = \frac{1}{8} \cdot \pi i f''(i/2) \text{ ; But } f''(z) = \pi^2 e^{\pi z}$$

$$\int_c \frac{e^{\pi z}}{(2z-i)^3} dz = \frac{\pi i}{8} \cdot \pi^2 e^{\pi i/2}$$

$$= \frac{\pi^3 i}{8} \left(\cos \pi/2 + i \sin \pi/2 \right)$$

$$= \frac{\pi^3 i}{8} \left(0 + i(1) \right)$$

$$= \frac{\pi^3 i^2}{8} = \frac{-\pi^3}{8} \qquad \because i^2 = -1$$

Thus $\int_C \frac{e^{\pi z}}{(2z-i)^3} dz = \frac{-\pi^3}{8}$

8) Evaluate $\int_C \frac{e^{2z}}{(z+1)^2 (z-2)} dz$ where $C : |z| = 3$

>> we shall first resolve $\frac{1}{(z+1)^2(z-2)}$ into p. fractions

Let $\frac{1}{(z+1)^2(z-2)} = \frac{A}{(z+1)} + \frac{B}{(z+1)^2} + \frac{C}{(z-2)}$

(or) $1 = A(z+1)(z-2) + B(z-2) + C(z+1)^2$

put $z = -1$, $B = -1/3$

$z = 2$ $\therefore C = 1/9$

$z = 0$ $\cdot$ $A = -1/9$

Now

$$\frac{1}{(z+1)^2 (z-2)} = -\frac{1}{9} \cdot \frac{1}{z+1} - \frac{1}{3} \cdot \frac{1}{(z+1)^2} + \frac{1}{9} \cdot \frac{1}{z-2}$$

x'ly by e^{2z} and integrating w.r.t z over C we have

$$\int_C \frac{e^{2z}}{(z+1)^2 (z-2)} dz = \frac{-1}{9} \int_C \frac{e^{2z}}{(z+1)} dz - \frac{1}{3} \int_C \frac{e^{2z}}{(z+1)^2} dz + \frac{1}{9} \int_C \frac{e^{2z}}{z-2} dz$$

① The points $z = a = -1$, $z = a = 2$ lies inside the cle $|z| = 3$

we shall consider Cauchy's integral formula in the forms

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a) \quad \text{and} \quad \int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(a).$$

taking $f(z) = e^{2z}$ we obtain $f'(z) = 2e^{2z}$

Now $\int_C \frac{e^{2z}}{z+1}\,dz = \int_C \frac{e^{2z}}{z-(-1)}\,dz = 2\pi i f(-1) = 2\pi i e^{-2} = \frac{2\pi i}{e^2}$

$$\int_C \frac{e^{2z}}{(z+1)^2}\,dz = \int_C \frac{e^{2z}}{[z-(-1)]^2}\,dz = \frac{2\pi i}{1!} f'(-1) = 2\pi i(2e^{-2})$$

i.e $\int_C \frac{e^{2z}}{(z+1)^2}\,dz = \frac{4\pi i}{e^2}$

Also $\int_C \frac{e^{2z}}{z-2} = 2\pi i f(2) = 2\pi i \cdot e^4$

Substituting these in the RHS of eqn (1)

$$\int_C \frac{e^{2z}}{(z+1)^2(z-2)}\,dz = -\frac{1}{9}\cdot\frac{2\pi i}{e^2} - \frac{1}{3}\cdot\frac{4\pi i}{e^2} + \frac{1}{9}\, 2\pi i e^4$$

$$= \frac{-7}{9}\,\frac{2\pi i}{e^2} + \frac{2\pi i}{9} e^4$$

Thus $\int_C \frac{e^{2z}}{(z+1)^2(z-2)}\,dz = \frac{2\pi i}{9}\left(e^4 - \frac{7}{e^2}\right)$

9) Evaluate $\int_C \frac{dz}{(z^2+4)^2}$ where $C: |z-i| = 2$, by cauchy's integral formula.

>> $C: |z-i| = 2$ is a circle with centre $(0,1)$ and radius 2.

we have $\frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2(z-2i)^2}$

Let $A = (0,1)$ be the centre and $r=2$ be the radius of C.

if $P_1 = (0,-2)$ and $P_2 = (0,2)$ then $AP_1 = 3 > 2$ and $AP_2 = 1 < 2$

Hence $(0,2)$ (or) $z = 2i$ only lies inside C.

we have cauchy's integral formula in the form

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz \quad \text{——— (1)}$$

Now $\frac{1}{(z^2+4)^2} = \frac{1}{[(z+2i)(z-2i)]^2} = \frac{1/(z+2i)^2}{(z-2i)^2}$.

Taking $f(z) = \frac{1}{(z+2i)^2}$ and $a = 2i$ we have

$$f'(z) = \frac{-2}{(z+2i)^3} \quad ; \quad f'(a) = f'(2i) = \frac{-2}{(4i)^3} = \frac{1}{32i}$$

(1) =>

$$\frac{1}{32i} = \frac{1}{2\pi i}\int_C \frac{1/(z+2i)^2}{(z-2i)^2}\,dz$$

ie $\frac{\pi}{16} = \int_C \frac{dz}{(z+2i)^2(z-2i)^2}$

Thus $\int_C \frac{dz}{(z^2+4)^2} = \frac{\pi}{16}$

10) Evaluate $\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)}\,dz$ where C is the circle (13)

i) $|z| = 3$, ii) $|z| = 1/2$ iii) $|z| = 3/2$

>> We shall first resolve $\frac{1}{(z-1)^2(z-2)}$ by P. fractions

Let $\frac{1}{(z-1)^2(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2} + \frac{C}{(z-2)}$ ----- (1)

(or) $1 = A(z-1)(z-2) + B(z-2) + C(z-1)^2$

put $z=1$ $\therefore B = -1$

$z=2$ $C = 1$

Eq. z^2 on. b. S $0 = A + C$ or $A = -C$ $A = -1$

Let $f(z) = \sin\pi z^2 + \cos\pi z^2$

x'ly (1) by $f(z)$ and int w.r.t z over C by using the value of the constants obtained we have.

$$I = \int_C \frac{f(z)}{(z-1)^2(z-2)}\,dz = -\int_C \frac{f(z)}{z-1}\,dz - \int_C \frac{f(z)}{(z-1)^2}\,dz + \int_C \frac{f(z)}{z-2}\,dz \quad \text{--- (2)}$$

$\Rightarrow I = I_1 + I_2 + I_3$ (say)

Case i) $C: |z| = 3$

The points $z=1$ and $z=2$ both lie within C.

Hence by Cauchy's integral formula.

$I_1 = -[2\pi i f(1)] = -2\pi i[\sin\pi + \cos\pi] = -2\pi i(0-1) = 2\pi i$

$I_2 = -[2\pi i f'(1)]$ But $f'(z) = 2\pi z(\cos\pi z^2 - \sin\pi z^2)$

Hence $I_2 = -[2\pi i \cdot 2\pi(\cos\pi - \sin\pi)] = 4\pi^2 i$

$I_3 = 2\pi i f(2) = 2\pi i [\sin 4\pi + \cos 4\pi] = 2\pi i(0+1) = 2\pi i$

Hence from (2),

$$I = 2\pi i + 4\pi^2 i + 2\pi i = 4\pi i + 4\pi^2 i$$

Thus $\underline{I = 4\pi i(1+\pi)}$ where $C: |z| = 3$

<u>case ii)</u> $C: |z| = 1/2$

The points $z=1$ and $z=2$ both lie outside C and hence $I_1 = 0 = I_2 = I_3$

Thus $I = 0$, where $C: |z| = 1/2$

<u>case iii)</u> $C: |z| = 3/2$

The points $z=1$ lies inside C and $z=2$ lies outside C.

Hence $I_1 = 2\pi i f(1) = 2\pi i$

$I_2 = 2\pi i f'(1) = 4\pi^2 i$

and $I_3 = 0$

Now $I = 2\pi i + 4\pi^2 i + 0$

$= 2\pi i(1+2\pi)$

Thus $\underline{I = 2\pi i(1+2\pi)}$

where $\underline{C: |z| = 3/2}$

11) Evaluate $\int_c \frac{\sin^6 z}{(z-\pi/6)^3} dz$ where c is the circle $|z|=1$ (14)

we have. $f^n(a) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}} dz$ ——— (1)

The point $z=a=\pi/6 \simeq 0.5$ lies with in the circle

$|z|=1$

Now by putting $n=2$ in (1) we have

$f^{(2)}(a) = f''(a) = \frac{2!}{2\pi i}\int_c \frac{f(z)}{(z-a)^3} dz$

taking $f(z)=\sin^6 z$ we have with $a=\pi/6$

$f''(\pi/6) = \frac{1}{\pi i}\int_c \frac{\sin^6 z}{(z-\pi/6)^3} dz$ ——— (2)

Consider $f(z) = \sin^6 z$

$\therefore f'(z) = 6\sin^5 z\cos z$;

$f''(z) = -6\sin^6 z + 30\sin^4 z\cos^2 z$

Now $f''(\pi/6) = -6\sin^6(\pi/6) + 30\sin^4(\pi/6)\cos^2(\pi/6)$

ie $f''(\pi/6) = -6\left(\frac{1}{2}\right)^6 + 30\left(\frac{1}{2}\right)^4\left(\frac{\sqrt{3}}{2}\right)^2$

$= -\frac{6}{64} + \frac{90}{64}$

$= \frac{84}{64}$

$= \frac{21}{16}$

Thus by pus this value in (2) we have

$\int_c \frac{\sin^6 z}{(z-\pi/6)^3} dz = \frac{21\pi i}{16}$

Singularity and Residue

* A point $z=a$ where $f(z)$ fails to be analytic is called a singularity or a singular point of $f(z)$.

* A point $z=a$ is called an isolated singularity of $f(z)$ if there exists a neighbourhood of a point 'a' which enclose no other singularities of $f(z)$.

Examples.

1) If $f(z) = \dfrac{z}{z-2}$ then $f(z)$ is not analytic at $z=2$ which is called the singular point of $f(z)$.

2) If $f(z) = \dfrac{z^2}{(z-1)(z+1)(z-2)}$ then the points $z=1$, $z=-1$, $z=2$ are all called singular points of $f(z)$.

It may be noted that the singular points of $f(z)$ are identified from the factors present in the denominator of $f(z)$ and the singular points are the points which make these factors zero.

Suppose $f(z)$ is expanded as a Laurent series about the point $z=a$ in the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n} \longrightarrow (1)$$

then the first term is called the analytic part of $f(z)$ and the 2nd term is called the principal part of $f(z)$. if the principal part of $f(z)$ consists of only a finite no of terms, say m, then we say that $z=a$. is a pole of order m, In particular a pole of order 1 $(m=1)$ is called a simple pole.

If the principal part of $f(z)$ at $z=a$ contain infinite no of terms then $z=a$ is called an essential singularity of $f(z)$. Also if the principal part of $f(z)$ is completely absent (ie $a_{-n}=0$) then $z=a$ is called a removable singularity of $f(z)$.

Examples.

1) If $f(z) = \dfrac{z^2}{(z-1)(z+1)^2(z-2)}$ then $z=1,2$ are poles of order 1 (simple poles) and $z=-1$ is a pole of order 2.

2) If $f(z) = \dfrac{e^z}{z^3(z^2+1)}$ then $z=0$ is a pole of order 3 and solving $z^2+1=0$ we get $z=\pm i$ which are simple poles.

3) If $f(z) = \dfrac{z+1}{(z^2+1)^2(4z^2-1)}$ then $z=\pm i$ are poles of order 2 and $z=\pm 1/2$ are simple poles,

Residues

The coefficient of $\frac{1}{z-a}$ that is a_{-1} in the expansion of $f(z)$ is called the residue of $f(z)$ at the pole $z=a$.

Formula for the residue at the pole.

If $z=a$ is a pole of order m of $f(z)$ then the residue of $f(z)$ at $z=a$ is denoted by $R[m,a]$ and is given by

$$R[m,a] = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\}$$

Cauchy's Residue Theorem.

Stmt:- If $f(z)$ is analytic inside and on the boundary of a simple closed curve C except for a finite number of poles $a, b, c, \ldots$ then the integral of $f(z)$ over C is equal to $2\pi i$ times the sum of the residues at the poles inside C. That is

$$\int_C f(z)\,dz = 2\pi i \left(a_{-1} + b_{-1} + c_{-1} + \cdots\right)$$

• working Procedure for problems to find $\int_C f(z)dz$ by using cauchy's residue theorem

→ we locate all the poles of $f(z)$ along with their orders by looking at the denominator of the given $f(z)$.

→ we identify the poles lying inside c.

→ we compute the residue for these poles using appropriate formula.

→ Finally we apply Cauchy's residue theorem

$$\int_C f(z)dz = 2\pi i \sum R$$

where $\sum R$ denote the sum of the residue at the poles lying in C.

1). Find the residues of the fun

$$f(z) = \frac{z}{(z+1)(z-2)^2} \text{ at i) } z=-1 \text{ ii) } z=2$$

>> $z=-1$ is a pole of order 1 (simple pole) and $z=2$ is a pole of order 2.

The residue of $f(z)$ for a pole of order m at $z=a$ is given by

$$R[m,a] = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\}$$

case i) Residue at $z=a=-1$ is given by $m=1$

$0!=1$

$$\lim_{z \to -1} \cancel{(z+1)} \cdot \frac{z}{\cancel{(z+1)}(z-2)^2}$$

$$= \lim_{z \to -1} \frac{z}{(z-2)^2} = \frac{-1}{(-1-2)^2} = \frac{-1}{(-3)^2} = \frac{-1}{9}$$

case ii) Residue at $z=a=2$ where $m=2$ is given by

$$\lim_{z \to 2} \frac{1}{1!} \frac{d}{dz}\left\{(z-2)^2 \frac{z}{(z+1)(z-2)^2}\right\}$$

$$= \lim_{z \to 2} \frac{d}{dz}\left(\frac{z}{z+1}\right)$$

$$= \lim_{z \to 2} \frac{(z+1)-z}{(z+1)^2} = \lim_{z \to 2} \frac{1}{(z+1)^2}$$

$$= \lim_{z \to 2} \frac{1}{(z+1)^2} = \frac{1}{9}$$

Thus the required residues are $-\frac{1}{9}$ and $\frac{1}{9}$

2) For the fun $f(z) = \dfrac{2z+1}{z^2-z-2}$ determine the poles and the residue at the poles.

>> In $f(z) = \dfrac{2z+1}{z^2-z-2} = \dfrac{2z+1}{(z-2)(z+1)}$

$z=2$, $z=-1$ are simple poles.

i) Residue at $z=a=2$ is given by

$$\lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}\cancel{(z-2)}\cdot\frac{2z+1}{\cancel{(z-2)}(z+1)}$$

$$= \lim_{z\to 2}\frac{2z+1}{z+1} = \frac{5}{3}$$

ii) Residue at $z=a=-1$ is given by

$$\lim_{z\to -1}(z+1)f(z) = \lim_{z\to -1}\cancel{(z+1)}\frac{2z+1}{(z-2)\cancel{(z+1)}}$$

$$= \lim_{z\to -1}\frac{2z+1}{(z-2)}$$

$$= \frac{2(-1)+1}{-1-2}$$

$$= \frac{-2+1}{-3}$$

$$= \frac{-1}{-3}$$

$$= \frac{1}{3}$$

Thus the residue at the poles are $5/3$ and $1/3$

3) Determine the residue at the pole of the fun $\frac{\sin z}{(2z-\pi)^2}$ (9) (18)

>> Let $f(z) = \frac{\sin z}{(2z-\pi)^2}$

Now $2z - \pi = 0$

$\Rightarrow 2z = \pi \Rightarrow z = \pi/2$

$\therefore z = a = \pi/2$ is a pole of order 2.

The residue of $f(z)$ at $z = a = \pi/2$ $(m=2)$ is given by

$$\lim_{z \to \pi/2} \frac{1}{1!} \frac{d}{dz} \left\{ (z - \pi/2)^2 \cdot \frac{\sin z}{(2z-\pi)^2} \right\}$$

$$= \lim_{z \to \pi/2} \frac{d}{dz} \left\{ \frac{\cancel{(2z-\pi)^2}}{2^2} \cdot \frac{\sin z}{\cancel{(2z-\pi)^2}} \right\}$$

$$= \lim_{z \to \pi/2} \frac{1}{4} \frac{d}{dz} (\sin z)$$

$$= \frac{1}{4} \lim_{z \to \frac{\pi}{2}} \cos z$$

$$= \frac{1}{4} \cos\left(\frac{\pi}{2}\right)$$

$$= \frac{1}{4} (0)$$

$$= 0$$

Thus the residue at the pole is 0.

4) Determine the residue at the poles for the fun $f(z) = \frac{z}{(z+1)^2 (z^2+4)}$

>> $z = -1$ is a pole of order 2.

Also, $(z^2+4) = 0 \Rightarrow (z+2i)(z-2i) = 0$

$\therefore$ $z = 2i, -2i$ are simple poles.

Let $R[m, a]$ denote the residue of $f(z)$ at $z = a$ for a pole of order m and we have

* $R[2,-1] = \frac{1}{1!} \lim_{z \to -1} \frac{d}{dz} \left\{ (z+1)^2 \frac{z}{(z+1)^2 (z^2+4)} \right\}$

$$= \lim_{z \to -1} \frac{d}{dz} \left\{ \frac{z}{z^2+4} \right\}$$

$$= \lim_{z \to -1} \frac{(z^2+4)1 - z(2z)}{(z^2+4)^2}$$

$$R[2,-1] = \lim_{z \to -1} \frac{4-z^2}{(z^2+4)^2} = \frac{4-1}{(1+4)^2} = \underline{\frac{3}{25}}$$

* $R[1, 2i] = \lim_{z \to 2i} (z-2i) \frac{z}{(z+1)^2 (z^2+4)}$

$$= \lim_{z \to 2i} \cancel{(z-2i)} \frac{z}{(z+1)^2 (z+2i) \cancel{(z-2i)}}$$

$$= \lim_{z \to 2i} \frac{z}{(z+1)^2 (z+2i)}$$

$$= \frac{2i}{(2i+1)^2 4i}$$

$$= \frac{1}{2} \frac{1}{4i^2+1+4i} = \frac{1}{2} \cdot \frac{1}{4i-3} \times \frac{4i+3}{4i+3}$$

$$= \frac{1}{2} \cdot \frac{4i+3}{(4i-3)(4i+3)} = \frac{1}{2} \cdot \frac{4i+3}{16i^2-9} \qquad i^2 = -1$$

$$R[1, 2i] = \underline{\frac{-1}{50} (4i+3)}.$$

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$$\text{Also } R[1,-2i] = \lim_{z\to -2i} \cancel{(z+2i)} \cdot \frac{z}{(z+1)^2 \cancel{(z+2i)}(z-2i)}$$

$$= \lim_{z\to -2i} \frac{z}{(z+1)^2(z-2i)}$$

$$= \frac{-2i}{(1-2i)^2(-4i)}$$

$$= \frac{1}{2}\cdot\frac{1}{1+4i^2-4i} = \frac{1}{2}\cdot\frac{1}{-3-4i}$$

$$= -\frac{1}{2}\cdot\frac{(3-4i)}{(3-4i)(3+4i)}$$

$$= -\frac{1}{2}\cdot\frac{3-4i}{25} = \frac{4i-3}{50}$$

$$R[1,-2i] = \frac{4i-3}{50}$$

5) Evaluate $\int_C \frac{e^{2z}}{(z+1)(z-2)}dz$ where C is the circle $|z|=3$

>> The poles of the fun $f(z) = \frac{e^{2z}}{(z+1)(z-2)}$ are $z=-1$, $z=2$ which are simple poles and both these lie within the circle $|z|=3$.

$\therefore$ residue of $f(z)$ at $z=a=-1$ is given by

$$\lim_{z\to -1}(z+1)f(z) = \lim_{z\to -1}(z+1)\frac{e^{2z}}{(z+1)(z-2)}$$

$$= \lim_{z\to -1}\frac{e^{2z}}{(z-2)} = \frac{e^{-2}}{-3} = \frac{-1}{3e^2} = R_1$$

Also residue of $f(z)$ at $z=a=2$ is given by

$$\lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}(z-2)\frac{e^{2z}}{(z+1)(z-2)}$$

$$= \lim_{z\to 2}\frac{e^{2z}}{z+1}$$

$$= \frac{e^4}{3} = R_2$$

we have Cauchy's Residue theorem

$$\int_c f(z)\,dz = 2\pi i\,[R_1 + R_2]$$

Thus $$\int_c \frac{e^{2z}}{(z+1)(z-2)}\,dz = 2\pi i\left(\frac{-1}{3e^2} + \frac{e^4}{3}\right)$$

$$= \frac{2\pi i}{3}\left(e^4 - \frac{1}{e^2}\right)$$

6) Evaluate $\int_c \frac{(z^2+5)}{(z-2)(z-3)}\,dz$ using residue theorem.

$C : |z| = 4$

>> The poles of the fu^n $f(z) = \frac{z^2+5}{(z-2)(z-3)}$ are $z=2$, $z=3$ and both the poles lie within the circle $|z|=4$

$\therefore$ residue at $z=2=a$ which is a simple pole is given by

$$\lim_{z\to 2} (z-2) f(z) = \lim_{z\to 2} \cancel{(z-2)} \frac{z^2+5}{\cancel{(z-2)}(z-3)}$$

$$= \lim_{z\to 2} \frac{z^2+5}{z-3}$$

$$= \frac{2^2+5}{2-3} = -9 = R_1$$

Similarly residue at $z = a = 3$ is given by

$$\lim_{z\to 3} (z-3) f(z) = \lim_{z\to 3} (z-3) \frac{z^2+5}{(z-2)(z-3)}$$

$$= \lim_{z\to 3} \frac{z^2+5}{z-2}$$

$$= \frac{3^2+5}{3-2} = 14 = R_2 \text{ (say)}$$

we have by Cauchy's residue theorem

$$\int_c f(z)\,dz = 2\pi i [R_1 + R_2]$$

$$= 2\pi i [-9 + 14]$$

$$= 2\pi i [5]$$

$$= 10\pi i$$

Thus $\int_c \frac{z^2+5}{(z-2)(z-3)} dz = 10\pi i$

7) Evaluate $\int_C \frac{dz}{z^3(z-1)}$ where C is the circle $|z|=2$.

>> Let $f(z) = \frac{1}{z^3(z-1)}$ and the poles of $f(z)$ are

$z=0$, $z=1$ Both the poles lie within $|z|=2$

$\therefore$ residue at $z=a=0$, being a pole of order 3 $(m=3)$ is given by

$$R_1 = \lim_{z\to 0} \frac{1}{(3-1)!} \frac{d^2}{dz^2}\left\{(z-0)^3 \frac{1}{z^3(z-1)}\right\}$$

$$= \lim_{z\to 0} \frac{1}{2} \frac{d^2}{dz^2}\left\{\frac{1}{z-1}\right\}$$

$$= \frac{1}{2} \lim_{z\to 0} \frac{2}{(z-1)^3} = \frac{-2}{2} = -1$$

Also residue at $z=a=1$, being a simple pole is given by

$$R_2 = \lim_{z\to 1} (z-1)\, f(z) = \lim_{z\to 1} \cancel{(z-1)} \cdot \frac{1}{z^3\cancel{(z-1)}}$$

$$= \lim_{z\to 1} \frac{1}{z^3}$$

$$= \frac{1}{1} = 1$$

$$\int_C f(z)dz = 2\pi i\,[R_1 + R_2]$$

$$= 2\pi i\,[-1+1]$$

$$= 0$$

Thus $\int_C \frac{dz}{z^3(z-1)} = 0$

8). Evaluate $\int_C \frac{e^{2z}}{(z+1)^4}\,dz$ where $C: |z|=3$ (21)

>> Let $f(z) = \frac{e^{2z}}{(z+1)^4}$

$z=-1$ is a pole of order 4 $(m=4)$ which lies inside $C: |z|=3$

$\therefore$ The residue of $f(z)$ at $z=a=-1$ is given by

$$\lim_{z\to -1} \frac{1}{(4-1)!} \frac{d^3}{dz^3}\left\{ \cancel{(z+1)^4} \frac{e^{2z}}{\cancel{(z+1)^4}} \right\}$$

$$= \lim_{z\to -1} \frac{1}{3!} \frac{d^3}{dz^3}\left\{ e^{2z} \right\}$$

$$= \lim_{z\to -1} \frac{1}{3!} \left(8e^{2z} \right).$$

$$= \frac{1}{6}\, 8e^{-2}$$

$$= \frac{4}{3}e^{-2}$$

Applying Cauchy's residue theorem we have

$$\int_C f(z)\,dz = 2\pi i\left[\frac{4}{3}e^{-2}\right] = \frac{8\pi i}{3e^2}$$

9) Using Cauchy's residue theorem evaluate

$\int_C \frac{z\cos z}{(z-\pi/2)^3}\,dz$ where $C: |z-1|=1$

>> Let $f(z) = \frac{z\cos z}{(z-\pi/2)^3}$ $\quad C: |z-1|=1$

here $z=\pi/2$ is a pole of order 3.

C is the ○le with centre at the point P (1, 0) and radius 1.

Let $z = \pi/2$ be the point $Q(\pi/2, 0)$

Distance $PQ = (\pi/2, -1) < 1$ and hence $z = \pi/2$ lies within the given ○le C

∴ The residue (R) at $z = \pi/2$ is given by

$$\lim_{z \to \pi/2} \frac{1}{(3-1)!} \frac{d^2}{dz^2} \left\{ (z-\pi/2)^3 \frac{z\cos z}{(z-\pi/2)^3} \right\}$$

$$R = \lim_{z \to \pi/2} \frac{1}{2} \frac{d^2}{dz^2} \{z\cos z\}$$

$$R = \lim_{z \to \pi/2} \frac{1}{2} \{-z\cos z - 2\sin z) = -1$$

hence Cauchy's Theorem → residue

$$\int_C f(z)\,dz = 2\pi i(R)$$

Thus $\int_C \frac{z\cos z}{(z-\pi/2)^3} dz = -2\pi i$

Bilinear Transformation (BLT)

The transformation $w = \frac{az+b}{cz+d}$, where, a, b, c, d are complex constants such that $ad-bc \neq 0$ is called a bilinear transformation.

Note:- Bilinear transformations preserve the cross-ratio of four points. z_1, z_2, z_3 and w_1, w_2, w_3

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Ex:- 1) Find the BLT that maps (transforms) the points $z_1 = 0$, $z_2 = -i$, $z_3 = -1$ on to the points $w_1 = i$, $w_2 = 1$, $w_3 = 0$

The required BLT is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

substitute z_1, z_2, z_3 and w_1, w_2, w_3

$$\frac{(w-i)(1-0)}{(w-0)(1-i)} = \frac{(z-0)(-i+1)}{(z+1)(-i-0)}$$

$$\frac{(w-i)}{w(1-i)} = \frac{z(1-i)}{(z+1)(-i)}$$

$$\frac{w-i}{w} = \frac{z}{z+1} \cdot \frac{(1-i)^2}{-i} = \frac{z}{z+1} \frac{[1+i^2-2i]}{-i}$$

$$\frac{w-i}{w} = \frac{z}{z+1} \cdot \frac{[1-1-2i]}{-i}$$

$$\frac{w-i}{w} = \frac{z}{z+1} \cdot \frac{-2i}{-i} = \frac{z}{z+1} \cdot 2 = \frac{2z}{z+1}$$

$$\frac{w-i}{w} = \frac{2z}{z+1}$$

$$(z+1)w-i = 2wz$$

$$wz+w-iz-i = 2wz$$

$$wz+w-2wz = iz+i$$

$$w-wz = i(1+z)$$

$$w[1-z] = i[1+z]$$

$$\boxed{w = \frac{i[z+1]}{-z+1}}$$ is the required transformation.

2) Find the bilinear transformation [BLT] that maps the points $z=-1, i, 1$ on the points $w=1, i, -1$ respectively, Find the fixed points of the transformation. (or) invariant points

>> $$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\therefore \frac{(w-1)(i+1)}{(w+1)(i-1)} = \frac{(z+1)(i-1)}{(z-1)(i+1)}$$

$$\frac{(w-1)}{(w+1)} = \frac{(z+1)}{(z-1)} \cdot \frac{(i-1)^2}{(i+1)^2} = \frac{z+1}{z-1} \cdot \frac{i^2+1-2i}{i^2+1+2i}$$

$$= \frac{(z+1)}{(z-1)} \cdot \frac{-1+1-2i}{-1+1+2i} = \frac{z+1}{z-1} \cdot \frac{-\cancel{2i}}{\cancel{2i}}$$

$$\frac{(w-1)}{(w+1)} = -\frac{(z+1)}{(z-1)}$$

$(w-1)(z-1) = -(z+1)(w+1)$

$wz - w - z + 1 = -[wz + z + w + 1]$

$wz - w - z + 1 = -wz - z - w - 1$

$wz - \cancel{w} - \cancel{z} + 1 + wz + \cancel{z} + \cancel{w} + 1 = 0$

$2wz + 2 = 0$

$2wz = -2$

$wz = -2/2 = -1$

$wz = -1$

$\boxed{w = -1/z}$ — ① is the required transforms.

For a fixed point, we have $w = z$

① ⟹ $z = -1/z$

$z^2 = -1$

$z = \pm i$

Thus $i, -i$ are its fixed points.

3) Find the BLT that transforms the points $z_1 = 1,\ z_2 = i,\ z_3 = -1$ onto the points $w_1 = 2,\ w_2 = i,\ w_3 = -2$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-2)(i+2)}{(w+2)(i-2)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{(w-2)}{(w+2)} = \frac{(z-1)}{(z+1)} \cdot \frac{(i+1)(i-2)}{(i-1)(i+2)} = \frac{z-1}{z+1} \cdot \frac{i^2-2i+i-2}{i^2+2i-i-2}$$

$$= \frac{(z-1)}{(z+1)} \cdot \frac{-1-i-2}{-1+i-2} = \frac{(z-1)}{(z+1)} \cdot \frac{-3-i}{-3+i}$$

$$= \frac{(z-1)}{(z+1)} \cdot \frac{-(3+i)}{-(3-i)}$$

$$\frac{w-2}{w+2} = \frac{(z-1)}{(z+1)} \cdot \frac{(3+i)}{(3-i)} = \rho$$

$$\frac{w-2}{w+2} = p$$

$$\Rightarrow w-2 = p(w+2)$$

$$w - 2 - pw - 2p = 0$$

$$w - wp - 2 - 2p = 0$$

$$w(1-p) = 2(1+p)$$

$$w\left[1 - \frac{(z-1)}{(z+1)}\cdot\frac{(3+i)}{3-i}\right] = 2\left[1 + \frac{(z-1)(3+i)}{(z+1)(3-i)}\right]$$

$$w\left[\frac{(z+1)(3-i) - (z-1)(3+i)}{''}\right] = 2\left[\frac{(z+1)(3-i) + (z-1)(3+i)}{''}\right]$$

$$w\left[\frac{3z - iz + 3 - i - (3z + iz - 3 - i)}{''}\right] = 2\left[\frac{3z - \cancel{iz} + \cancel{3} - i + 3z + \cancel{iz} - \cancel{3} - i}{''}\right]$$

$$w\left[\frac{\cancel{3z} - iz + 3 - \cancel{i} - \cancel{3z} - iz + 3 + \cancel{i}}{''}\right] = 2\left[\frac{6z - 2i}{(z+1)(3-i)}\right]$$

$$w\left[\frac{(-2iz + 6)}{(z+1)(3-i)}\right] = 2\left[\frac{(6z-2i)}{(z+1)(3-i)}\right]$$

$$w = \frac{(6z-2i)}{(6-2iz)}\cdot\frac{(z+1)\cancel{(3-i)}}{(z+1)(3-i)}$$

$$\boxed{w = \frac{6z-2i}{6-2iz}} \Rightarrow \frac{2(3z-i)}{2(3-iz)}$$

$$\boxed{w = \frac{(3z-i)}{3-iz}}$$ is the required

BLT

HW 1, i, -1 onto the points i, 0, -i respectively
z also find fixed points.   w

4) Find the BLT which maps the points 0, 1, ∞ onto the points $-5, -1, 3$ respectively. (8/24)

here $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$, so that $\frac{1}{z_3} = 0$

and $w_1 = -5$, $w_2 = -1$, $w_3 = 3$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(z-z_1)\left(\frac{z_2}{z_3}-1\right)}{\left(\frac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\frac{(w+5)(-1-3)}{(w-3)(-1+5)} = \frac{(z-0)\left(\frac{1}{\infty}-1\right)}{\left(\frac{z}{\infty}-1\right)(1-0)}$$

$$\frac{(w+5)(-4)}{(w-3)(4)} = \frac{z(0-1)}{(0-1)(1-0)} = \frac{-z}{-1} = z$$

$$\frac{(w+5)}{(w-3)} = \frac{4z}{-4} = -z$$

Note $\frac{0}{\text{anything}} = 0$

$\frac{\text{anything}}{0} = \infty$

$(w+5) = -[z(w-3)]$

$(w+5) = -[wz-3z]$

$w+5 = -wz+3z$

$w+5+wz-3z = 0$

$w+wz+5-3z = 0$

$w+wz-3z+5 = 0$

~~$w[1+z-3] = -5$~~

$w[1+z] = 3z-5$

$$\therefore w = \frac{3z-5}{1+z} = \frac{3z-5}{z+1}$$ is the required **BLT**

5) Find the BLT which maps the points $z = 0, i, \infty$ onto the points $w = 1, -i, -1$ respectively. Find the invariant points.

Here $z_1 = 0, \quad z_2 = i, \quad z_3 = \infty$ so that $\frac{1}{z_3} = \frac{1}{\infty} = 0$

$w_1 = 1, \quad w_2 = -i, \quad w_3 = -1$

Hence $$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)\cdot z_3\left(\frac{z_2}{z_3}-1\right)}{z_3\left(\frac{z}{z_3}-1\right)(z_2-z_1)} = \frac{(z-z_1)\left(\frac{z_2}{z_3}-1\right)}{\left(\frac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\frac{(w-1)(-i+1)}{(w+1)(-i-1)} = \frac{(z-0)\left(\frac{i}{\infty}-1\right)}{\left(\frac{z}{\infty}-1\right)(i-0)} = \frac{z(0-1)}{(0-1)(i-0)} = \frac{-z}{-i}$$

$$\frac{(w-1)}{(w+1)} = \frac{(-i-1)}{(-i+1)}\cdot\frac{z}{i} = \frac{-iz-z}{-i^2+i} = \frac{-iz-z}{1+i}$$

$$= \frac{-z(i+1)}{(i+1)} = -z$$

$$\frac{(w-1)}{(w+1)} = -z$$

$(w-1) = -z[w+1]$

$w-1 = -wz-z$

$w-1+wz+z=0$

$w-1+wz+z=0$

$w+wz+z-1=0$

$w[1+z] = 1-z$

$$\boxed{w = \frac{1-z}{1+z}}$$ is the required BLT

To find the fixed points

$w = z$

$z(z+1) = 1-z$

$z^2+z-1+z=0$

$z^2+2z-1=0$

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$= \frac{-2\pm\sqrt{4+4}}{2}$

$= \frac{-2\pm\sqrt{8}}{2}$

$= \frac{-2\pm\sqrt{4\times2}}{2}$

$= \frac{-2\pm2\sqrt{2}}{2}$

$= -1\pm\sqrt{2}$

$-1+\sqrt{2}$ & $-1-\sqrt{2}$ are the fixed points

Find the BLT that transforms the points $z_1 = i$, $z_2 = 1$, $z_3 = -1$ onto the points $w_1 = 1$, $w_2 = 0$, $w_3 = \infty$ respectively.

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\, w_3(w_2/w_3-1)}{w_3(w/w_3-1)\,(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)(w_2/w_3-1)}{(w/w_3-1)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-1)(0/\infty-1)}{(w/\infty-1)(0-1)} = \frac{(z-i)(1+1)}{(z+1)(1-i)}$$

$$\frac{(w-1)(0-\cancel{1})}{(0-1)(0-1)} = \frac{(z-i)\,2}{(z+1)(1-i)}$$

$$\frac{(w-1)(-1)}{(-1)} = \frac{(z-i)\,2}{(z+1)(1-i)}$$

$$(w-1) = \frac{-2(z-i)}{(z+1)(1-i)}$$

$$w = 1 - \frac{2(z-i)}{(z+1)(1-i)} = \frac{(z+1)(1-i) - 2(z-i)}{(z+1)(1-i)}$$

$$= \frac{\cancel{z} - iz + 1 - \cancel{i} - 2z + \cancel{2i}}{(z+1)(1-i)}$$

$$= \cancel{\frac{-z-3i-iz+1}{(z+1)(1-i)}} = \frac{-z+i+1-iz}{''}$$

$$= \cancel{-z(1+i)} \qquad = \frac{-z-iz+(1+i)}{''}$$

$$= \frac{-z(1+i)+(1+i)}{''}$$

$$= \frac{(1+i)(1-z)}{''}$$

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$\times^{ly}$ and $\div$ by $(1+i)$

$$w = \frac{(1+i)^2(1-z)}{(1+i)(1-i)(1+z)}$$

$$w = \frac{[1-1+2i](1-z)}{[1-i^2](1+z)}$$

$$= \frac{2i(1-z)}{2(1+z)}$$

$$= i\frac{(1-z)}{(1+z)}$$ is the required transformation.

H.W

(*) Find the bilinear BLT which has 1 and i as fixed points and which maps to 0 to -1.

>> Since 1 and i are fixed points of the required transformation. the required transformation maps the points $z_1 = 1$ and $z_2 = i$ to the points $w_1 = 1$, $w_2 = i$ respectively. Also, it is given that the transformation maps the points $z_3 = 0$ to the point $w_3 = -1$. Thus the required transformation maps to points $z_1 = 1$, $z_2 = i$ & $z_3 = 0$ To the points $w_1 = 1$, $w_2 = i$ and $w_3 = -1$.

$\times^{ly}$ & $\div$ by $(1+i)$

Ans. $\dfrac{(1+2i)z - i}{2+i}$

Discussion of Conformal Transformations.

Given the transformation $w=f(z)$, we put $z=x+iy$ (or) $z=re^{i\theta}$ to obtain u and v as functions of x, y (or) r, θ we find the image in w-plane corresponding to the given curve in the z-plane. Some times we need to make some judicious elimination from u and v for obtaining the image in the w-plane.

1) Discussion of $w = e^{z}$

(OR)

Show that the transformation $w=e^{z}$ map straight lines parallel to the co-ordinate axes in the z-plane into orthogonal trajectories in the w-plane and sketch the region.

Proof :- Consider $w = e^{z}$

ie $u+iv = e^{x+iy}$

$$= e^{x} e^{iy}$$

$$= e^{x} (\cos y + i \sin y) \quad \because e^{i\theta}$$

$$= e^{x}\cos y + i e^{x} \sin y$$

$\therefore \; u = e^{x}\cos y$ and $v = e^{x}\sin y$ ——— (1)

Separating the Re and Im parts

we shall find the image in the w-plane corresponding to the straight lines parallel to the co-ordinate axes in the z-plane.
ie $x = \text{constant}$ and $y = \text{constant}$.

Let us eliminate x and y separately from (1)

Squaring and adding we get

$$u^2+v^2 = e^{2x} \quad \text{——— (2)}$$

Also by dividing we get

$$\frac{v}{u} = \frac{e^x \sin y}{e^x \cos y} = \tan y \quad \text{——— (3)}$$

Case i) Let $x = c_1$ where c_1 is a constant.

Eqn (2) $\Rightarrow$

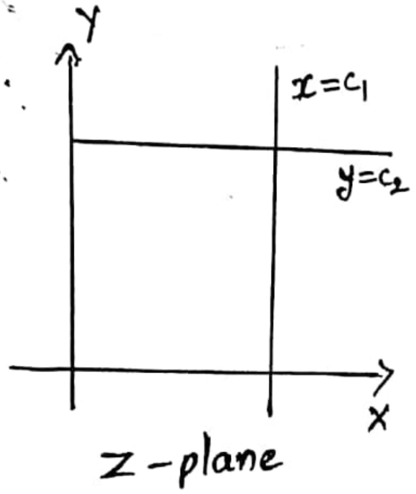
$$u^2+v^2 = e^{c_1} = \text{constant} = r^2$$

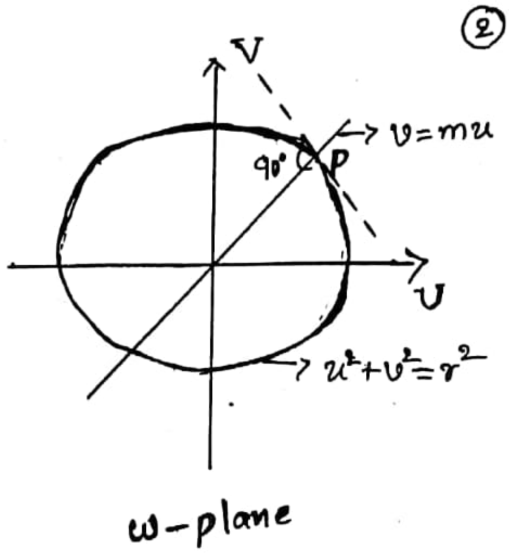
ie $u^2+v^2=r^2$ represents a circle with centre origin and radius r in the w-plane.

Case ii) Let $y = c_2$ where c_2 is a constant.

Eqn (3) $\Rightarrow$ $\frac{v}{u} = \tan c_2 = m$

$\therefore$ $v = mu$ represents a straight line passing through the origin in the w-plane.





Conclusion: The straight line parallel to the x-axis ($y=c_2$) in the z-plane maps onto a st line passing through the origin in the w-plane. The st line parallel to y-axis ($x=c_1$) in the z-plane maps onto a circle with centre origin and radius r where $r=e^{c_1}$ in the w-plane.

Suppose we draw a tangent at the point of intersection of these two curves in the w-plane (ie, at P as in the above fig) the angle subtended is equal to 90°. Hence these two curves can be regarded as orthogonal trajectories of each other.

2) Discussion of $w = z^2$

(OR)

Find the images in the w-plane corresponding to the straight lines $x = c_1$, $x = c_2$; $y = k_1$, $y = k_2$, under the transformation $w = z^2$. Indicate the region with sketches.

Proof:- Consider $w = z^2$

ie $u + iv = (x+iy)^2$

$= x^2 + (iy)^2 + 2(xiy)$ but $i^2 = -1$

$= (x^2 - y^2) + i(2xy)$

ie $u = (x^2 - y^2)$ and $v = 2xy$ —— (1)

Case i) Let us consider $x = c_1$, c_1 is a constant.

The set of equations (1) $\Rightarrow$

$u = c_1^2 - y^2$; $v = 2c_1 y$

Now $y = v/2c_1$ and substituting this in u

$u = c_1^2 - (v^2/4c_1^2)$

(or) $v^2/4c_1^2 = c_1^2 - u$

(or) $v^2 = -4c_1^2(u - c_1^2)$

This is a parabola in the w-plane symmetrical about the real axis with its vertex at $(c_1^2, 0)$

and focus at the origin. It may be observed that the line $x=-c_1$ is also transformed into the same parabola.

Case ii) Let us consider $y=c_2$, c_2 is a Constant.

The set of Eqns (1) =>

$$u=x^2-c_2^2, \quad v=2xc_2$$

Now $x=v/2c_2$ and substituting this in u

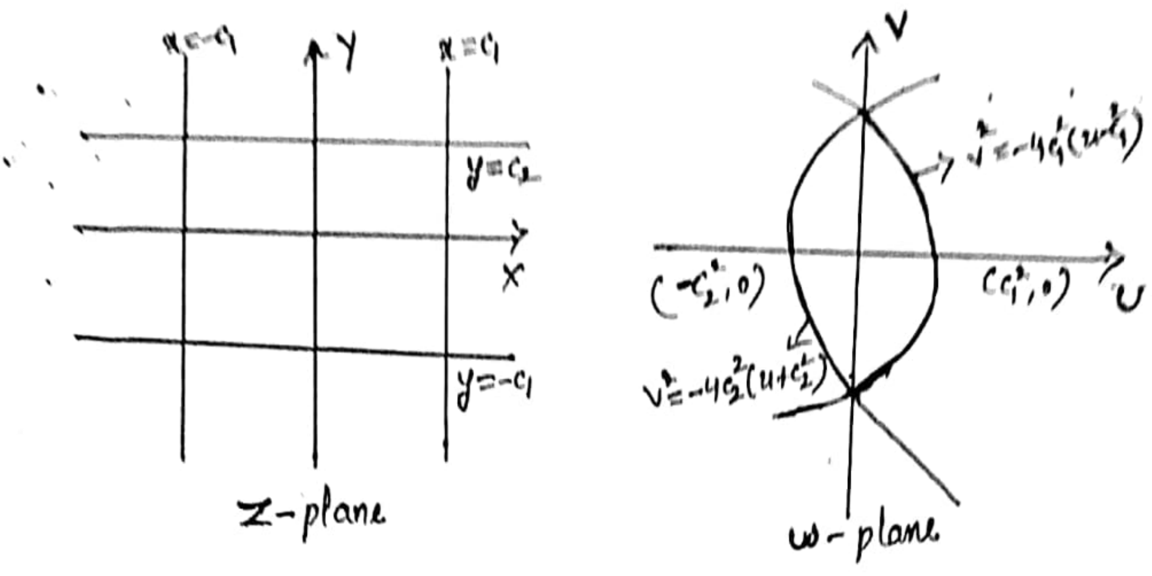
$$u=(v^2/4c_2^2)-c_2^2$$

(or) $$v^2/4c_2^2=u+c_2^2$$

(or) $$v^2=4c_2^2(u+c_2^2)$$

This is also a parabola in the w-plane symmetrical about the real axis whose vertex is at $(-c_2^2,0)$ and focus at the origin. Also the line $y=-c_2$ is transformed into the same parabola.

Hence from these two cases we conclude that the St-lines parallel to the co-ordinate axes in the z-plane map onto parabolas in the w-plane.



3) Discussion of $w = z + \left(\frac{1}{z}\right)$, $z \neq 0$

Consider the transformation

$$w = z + \frac{1}{z} \quad \text{——— (1)}$$

Here $f'(z) = 1 - \frac{1}{z^2}$, From this, we note that $f'(z)$ exists and not zero when $z \neq 0$ and $z^2 \neq 1$.

$\therefore$ the transformation ① is conformal at all points except at '0' and '±1'. This transformation is known as the Joukowski's transformation.

Taking $z = re^{i\theta}$ in ① we obtain

$$u + iv = re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$$u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$\therefore \quad u = \left(r + \frac{1}{r}\right)\cos\theta , \quad v = \left(r - \frac{1}{r}\right)\sin\theta \quad \text{—— ②}$$

④

∴ From ellipse we get

$$\frac{u^2}{(r+1/r)^2} + \frac{v^2}{(r-1/r)^2} = \cos^2\theta + \sin^2\theta = 1 \quad \text{——— (3)}$$

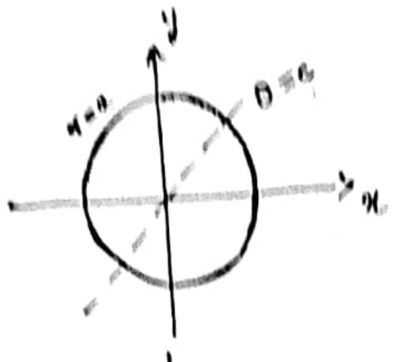
Consider the polar eqⁿ $r=a\,(\neq 1)$, a constant, which represents a circle centred at the origin in the z-plane. Then eqⁿ (3) represents an ellipse having centre at the origin of the w-plane and u and v-axes as it's axes.

Thus, under the transformation ① the circle $r=a$ centred at the origin in the z-plane is transformed into the ellipse ③ in the w-plane.

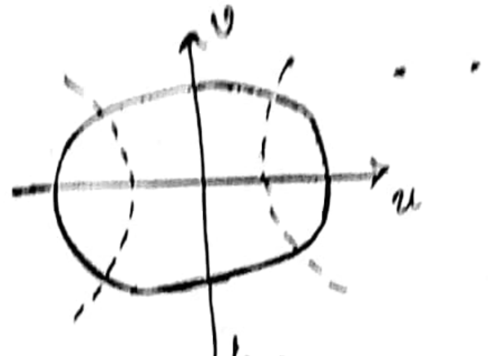
From relations ①, we also obtain

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(r+\frac{1}{r}\right)^2 - \left(r-\frac{1}{r}\right)^2 = 4 \quad \text{——— (4)}$$

For $\theta = c$, a constant, eqⁿ ④ represents a hyperbola having centre at the origin of the w-plane and the u-axis and v-axis as axes. Thus under the transformation ① the radial line $\theta = c$ in the z-plane is transformed to the hyperbola ④ in the w-plane.



z-plane



w-plane

a takes different constant values, the eqn $r = a$ represents a family of concentric circles in the z-plane and eqn (1) represents a family of ellipses in the w-plane all of which have the origin as their centre and u- and v- axes as their axes. Thus under the transformation (1), a family of concentric circles having their centres at the origin in the z-plane transform to the family of concentric and coaxal ellipses having their centres at the origin in the w-plane.