

MODULE -2

Time Domain Analysis of Control systems.

Time response :-



The response given by the system which is a function of time to applied excitation (input) is called as time response.

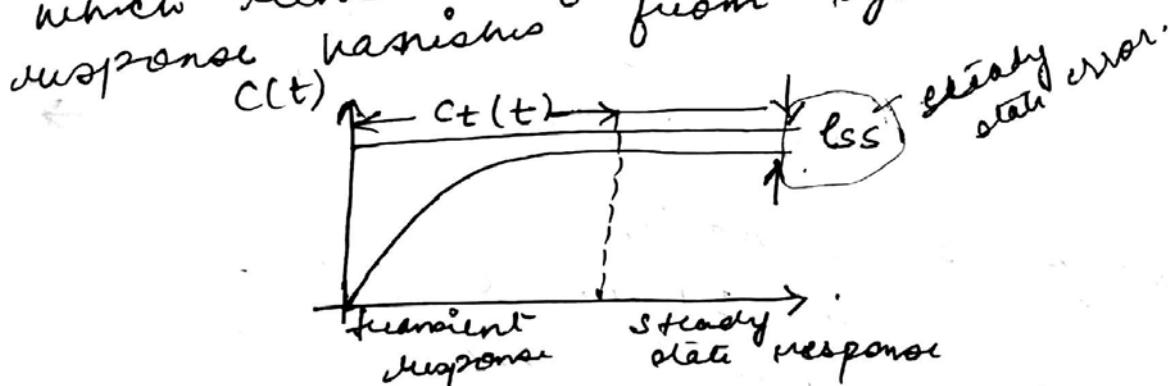
Transient Response :-

The O/P variation during the time it takes to achieve its final value is called transient response.

The time required to achieve the final value is called transient period.

Steady State response :-

It is that part of time response which remains after complete transient response vanishes from system output.



Steady State error :

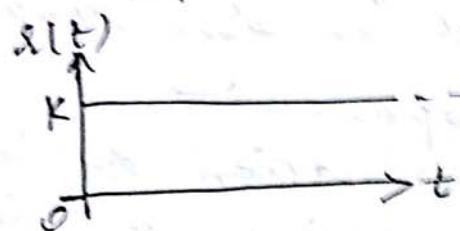
The difference b/w the desired O/P and the actual O/P of the system is called steady state error which is denoted as ess . This error indicates accuracy and play a important role.

in designing the system.

Some important functions :-

i) Step Function : A step function represents an instantaneous change in the reference input. The step function is defined as

$$x(t) = k, \quad t \geq 0 \quad \underline{\text{or}} \quad x(t) = k u(t)$$
$$0, \quad t < 0.$$



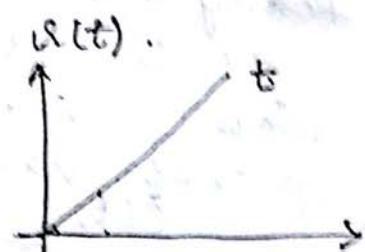
ii) Ramp Function : The ramp function is a function that changes constantly with time.

The mathematical description of a ramp signal is as follows -

$$x(t) = \begin{cases} kt, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

or

$$x(t) = kt u(t).$$



iii) Parabolic Function :-

The parabolic function represents a signal i.e. one order faster than ramp signals.

$$x(t) = \begin{cases} \frac{kt^2}{2}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

or

$$x(t) = \frac{kt^2}{2} u(t).$$

Test signal	$u(t)$	$R(s)$.
i) Step	$u(t) = \begin{cases} K, & t \geq 0 \\ 0, & t < 0 \end{cases}$	$R(s) = \frac{k}{s}$.
ii) Ramp.	$u(t) = \begin{cases} kt, & t \geq 0 \\ 0, & t < 0 \end{cases}$	$R(s) = \frac{k}{s^2}$.
iii) Parabolic	$u(t) = \begin{cases} \frac{Kt^2}{2}, & t \geq 0 \\ 0, & t < 0 \end{cases}$	$R(s) = \frac{k}{s^3}$.

First order system
The first order system without zeros can be described by the transfer function $G(s) = \frac{a}{s+a}$.

If the i/p is unit step we have

$$R(s) = \frac{1}{s}$$

Then the o/p of the system is given by

$$\frac{C(s)}{R(s)} = G(s).$$

$$C(s) = R(s) G(s)$$

$$C(s) = \frac{1}{s} \times \frac{a}{s+a}$$

$$C(s) = \frac{a}{s(s+a)}$$

$$C(s) = \frac{A}{s} + \frac{B}{s+a}$$

put $s=0$

$$a = A(s+a) + B(s)$$

$$a = A(a) + B(0)$$

$$\boxed{A=1}$$

Put $s=-a$.

$$a = \frac{A(0) + B(-a)}{B^2 - 1}$$

$$C(s) = \frac{1}{s} - \frac{1}{s+a}$$

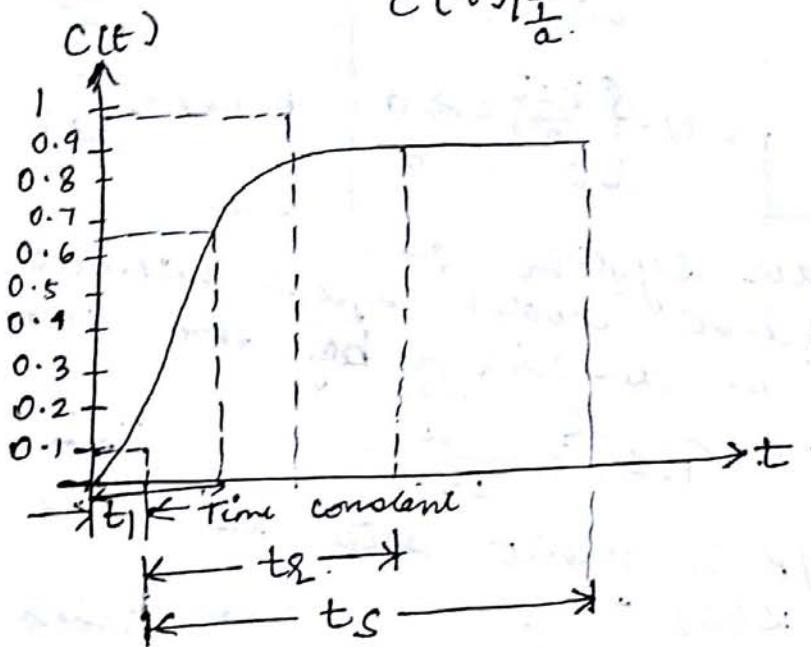
Taking inverse Laplace Transform
on both sides.

$$C(t) = 1 - e^{-at}, t \geq 0.$$

when $t = \frac{1}{a}$, $C(t) = 1 - e^{-\frac{1}{a}}$.

$$C(t) = 1 - e^{-\frac{1}{a}}$$

$$C(t) \approx 0.63.$$



The time constant is the time taken by the step response to rise to 63% of its final value. The reciprocal of time constant has the units of frequency. Thus $\frac{1}{a}$ is the time constant and a is the frequency.

ii) Rise time "tr": It is defined as the time for the waveform to go from 0.1 to 0.9 of its final value.

$$C(t_1) = 0.1$$

$$C(t_2) = 0.9$$

$$C(t) = 1 - e^{-at}$$

$$\text{at } t = t_1, C(t_1) = 1 - e^{-at_1}$$

$$(0.1) = 1 - e^{-at_1}$$

$$t_1 = \frac{0.1}{a}$$

at $t = t_2$

$$C(t_2) = 1 - e^{-at_2}$$

$$0.9 = 1 - e^{-at_2}$$

$$t_2 = \frac{2.3}{a}$$

$$t_s = t_2 - t_1 = \frac{2.3}{a} - \frac{0.1}{a}$$

$$= \frac{2.2}{a}$$

(iii) Settling time : The settling time for step response is defined as the time to reach and stay within $\pm\%$ of its final value.

$$C(t) = 98\%$$

$$= 0.98$$

w.k.t $C(t) = 1 - e^{-at}$

$$0.98 = 1 - e^{-at}$$

$$t_s = \frac{4}{a}$$

Find the step response $C(t)$ for the system described by $G(s) = \frac{4}{s+4}$. Also find the rise time and settling time.

$$G(s) = \frac{4}{s+4}, R(s) = \frac{1}{s}$$

$$C(s) = G(s)R(s)$$

$$C(s) = \frac{4}{s(s+4)}$$

$$C(s) = \frac{A}{s} + \frac{B}{s+4}$$

$$C(s) = \frac{1}{s} - \frac{1}{s+4}$$

Taking Inverse Laplace transform

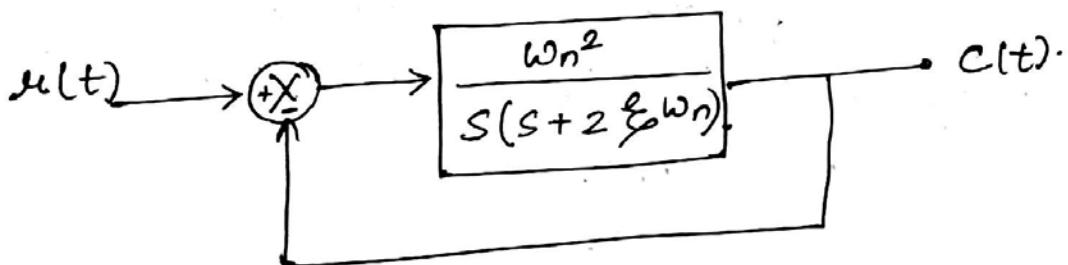
$$C(t) = 1 - e^{-4t}, t \geq 0$$

Time constant $T = \frac{1}{\alpha} = 0.25 \text{ sec}$

Rise time, $t_r = \frac{2.2}{\alpha} = \frac{2.2}{4} = 0.55 \text{ sec}$

Setting time, $t_s = \frac{4}{\alpha} = \frac{4}{4} = 1 \text{ sec}$

Time Response analysis of II order control system
consider a II order system shown below:-



Since the highest power of s in the denominator is 2 ∴ the above system is of order 2.

Closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$\frac{C(s)}{R(s)} = \frac{\frac{w_n^2}{s(s + 2\xi w_n)}}{1 + \frac{w_n^2}{s(s + 2\xi w_n)}} \cdot 1$$

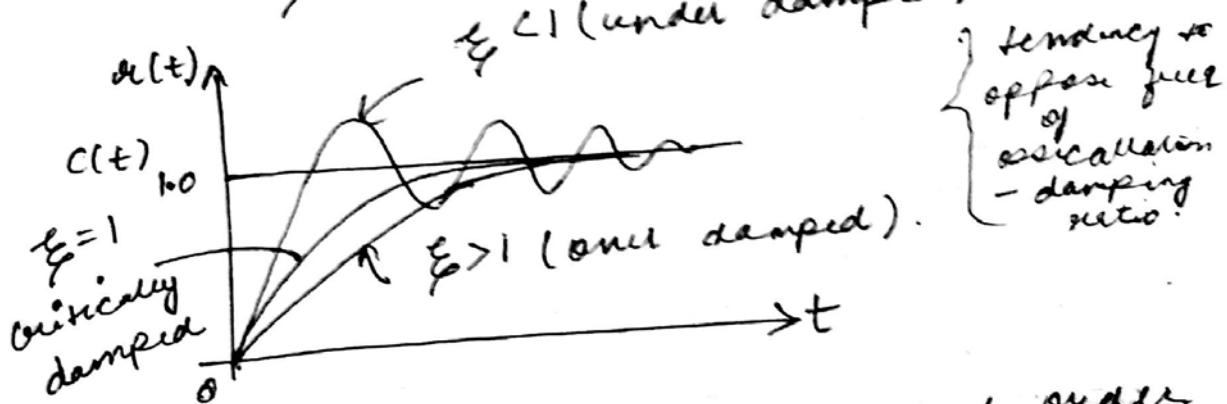
$$\frac{C(s)}{R(s)} = \frac{w_n^2}{s(s + 2\xi w_n)} \times \frac{s(s + 2\xi w_n)}{s(s + 2\xi w_n) + w_n^2}$$

$$\frac{C(s)}{R(s)} = \frac{w_n^2}{s(s + 2\xi w_n) + w_n^2}$$

$$\frac{C(s)}{R(s)} = \frac{w_n^2}{s^2 + 2\xi w_n s + w_n^2}$$

where ω_n = natural frequency of oscillation
 ξ = damping factor

- $\xi < 1$ under damped oscillations
- $\xi = 1$ critically damped oscillations
- $\xi > 1$ over damped oscillations.



Time response analysis of a second order system to a unit step input.

Consider a closed loop transfer function of a second order system given by:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$C(s) = R(s) \cdot \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

But $R(s) = \frac{1}{s}$ (For step i/p).

$$C(s) = \frac{\omega_n^2}{\frac{s}{s^2 + 2\xi\omega_n s + \omega_n^2}}$$

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

Resolving into partial fraction:-

$$C(s) = \frac{A}{s} + \frac{BS + C}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\omega_n^2 = A(s^2 + 2\xi\omega_n s + \omega_n^2) + (Bs + c)(s)$$

Comparing coefficient of ω_n^2

$$1 = A$$

Comparing coefficient of s^2

$$0 = A + B$$

$$0 = 1 + B$$

$$\boxed{B = -1}$$

Comparing coefficient of s

~~$$0 = A + C$$~~

$$0 = A 2\xi\omega_n s + Cs$$

$$A 2\xi\omega_n + C = 0$$

$$(1) 2\xi\omega_n + C = 0$$

$$\boxed{C = -2\xi\omega_n}$$

$$C(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{(s + 2\xi\omega_n)}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Consider $s^2 + 2\xi\omega_n s + \omega_n^2$ and it can be represented as -

$$\Rightarrow s^2 + 2\xi\omega_n s + \omega_n^2 + (\xi\omega_n)^2 - (\xi\omega_n)^2$$

[Add & subtract $(\xi\omega_n)^2$]

$$\Rightarrow (s + \xi\omega_n)^2 + \omega_n^2 (1 - \xi^2)$$

$$\Rightarrow (s + \xi\omega_n)^2 + \omega_d^2$$

where ω_d = Frequency of damped oscillation given by

$$\omega_d = \omega_n \sqrt{1 - \xi^2}$$

$S + 2\xi\omega_n$ is represented as -

$$S + \xi\omega_n + \xi\omega_n.$$

$$\therefore C(S) = \frac{1}{S} - \frac{(S + \xi\omega_n)}{(S + \xi\omega_n)^2 + \omega_d^2} - \frac{\xi\omega_n}{(S + \xi\omega_n)^2 + \omega_d^2}$$

Taking Inverse Laplace transform on both sides -

$$L \left[\frac{S+b}{(S+b)^2 + a^2} \right] = e^{-bt} \cos at$$

$$L \left[\frac{1}{(S+b)^2 + a^2} \right] = \frac{1}{a} e^{-bt} \sin at.$$

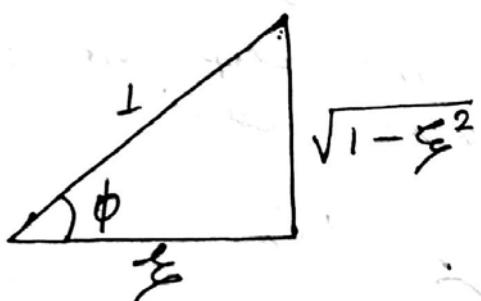
$$\therefore C(t) = 1 - e^{-\xi\omega_n t} \cos \omega_d t - \xi\omega_n \cdot \frac{1}{\omega_d} e^{-\xi\omega_n t} \sin \omega_d t.$$

$$C(t) = 1 - e^{-\xi\omega_n t} \cdot \left[\cos \omega_d t + \frac{\xi\omega_n}{\omega_n \sqrt{1 - \xi^2}} \cdot \sin \omega_d t \right].$$

$$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\omega_n \sqrt{1 - \xi^2}} \left[\omega_n \sqrt{1 - \xi^2} \cos \omega_d t + \xi\omega_n \sin \omega_d t \right] \quad (i)$$

$$\sin(\omega_d t + \phi) = \sin \omega_d t \cos \phi + \cos \omega_d t \sin \phi.$$

Comparing the above eqn with eqn (i) -



$$\sin \phi = \frac{\sqrt{1 - \xi^2}}{1} = \sqrt{1 - \xi^2}$$

$$\cos \phi = \frac{\xi}{1}$$

$$\tan \phi = \frac{\sqrt{1 - \xi^2}}{\xi}$$

$$\phi = \tan^{-1} \frac{\sqrt{1 - \xi^2}}{\xi}$$

$$c(t) = 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1 - \xi^2}} \left[\sin \phi \cos \omega_n t + \cos \phi \sin \omega_n t \right]$$

$$c(t) = 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1 - \xi^2}} \left[\sin(\omega_n t + \phi) \right] \quad \text{--- (ii)}$$

Steady state Error is given by

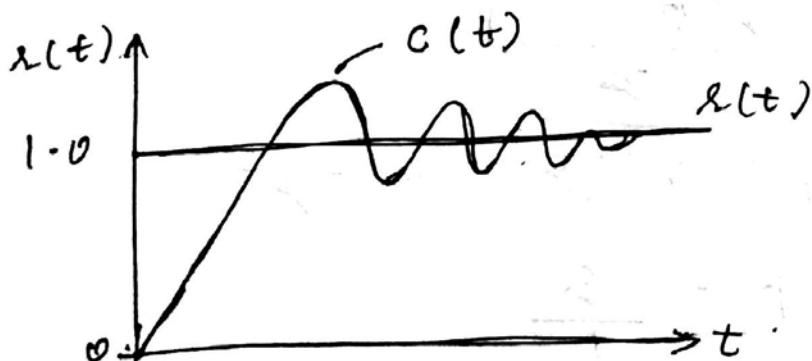
$$e_{ss} = \lim_{t \rightarrow \infty} e(t)$$

$$= \lim_{t \rightarrow \infty} [r(t) - c(t)]$$

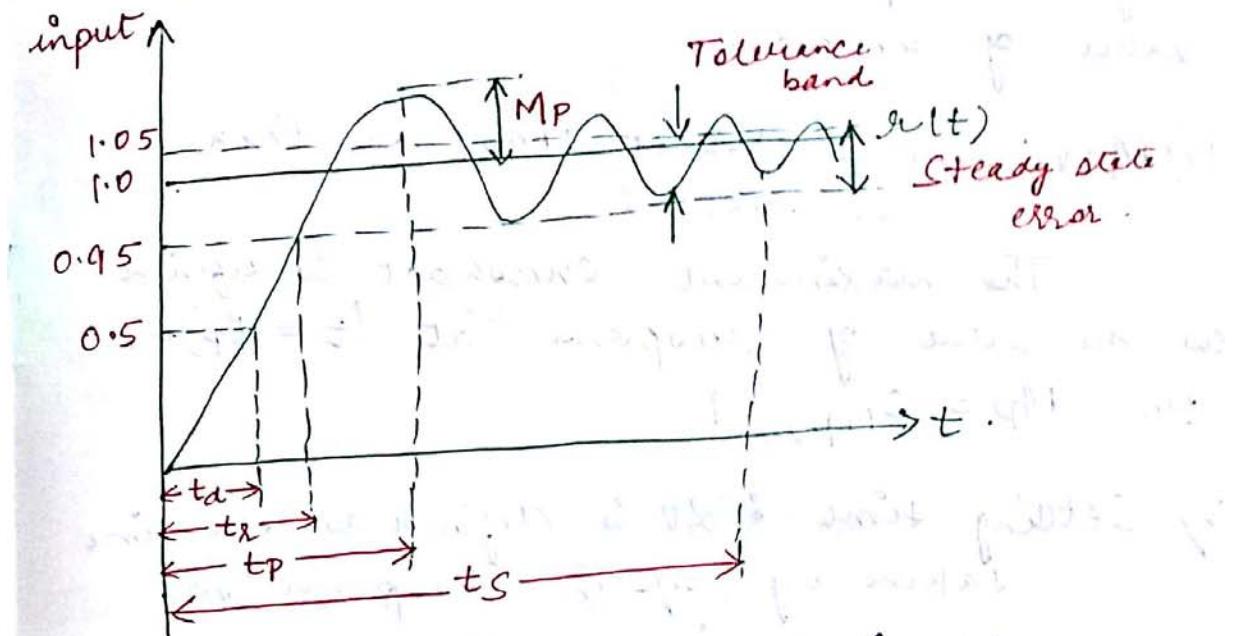
$$e_{ss} = \lim_{t \rightarrow \infty} \left[1 - 1 + \frac{e^{-\xi \omega_n t}}{\sqrt{1 - \xi^2}} (\sin \omega_n t + \phi) \right]$$

$$\boxed{e_{ss} = 0}$$

Since in steady state error is zero so
Second order system is capable of
responding to a unit step I/P without
any error as shown below -



Time Domain specification of a II-order system of a unit step i/P :-



The various specifications are :-

- i) Delay time (t_d)
- ii) Rise time (t_r)
- iii) Peak time (t_p)
- iv) Maximum overshoot (M_p) or overall peak overshoot
- v) Settling time (t_s)
- vi) Steady state error.

i) Delay time (t_d): The time taken by the system response to reach 50% of its final steady state value for the first time.

ii) Rise time (t_r): It is defined as the time taken by the response to change either from 10% to 90% or from 5% to 95% or overall from 0 to 100% of its final steady

Stati value for the first time.

iii) Peak time (t_p): The time taken by the system response to reach the maximum value of response.

iv) Maximum overshoot (M_p) or Peak overshoot:

The maximum overshoot is defined as the value of response at $(t = t_p) - 1$

$$\text{or } M_p = e_{(t_p)} - 1$$

v) Settling time : It is defined as the time taken by system response to reach a particular allowable tolerance band and to stay within it. Normally, 2% tolerance band is used.

vi) Steady State Error : It is defined as the difference b/w the desired O/P & actual O/P as time tends to infinity.

$$e_{ss} = \lim_{t \rightarrow \infty} [x(t) - C(t)]$$

Mathematical Expression for rise time(t_r), peak time(t_p), peak overshoot (M_p) and Settling time.

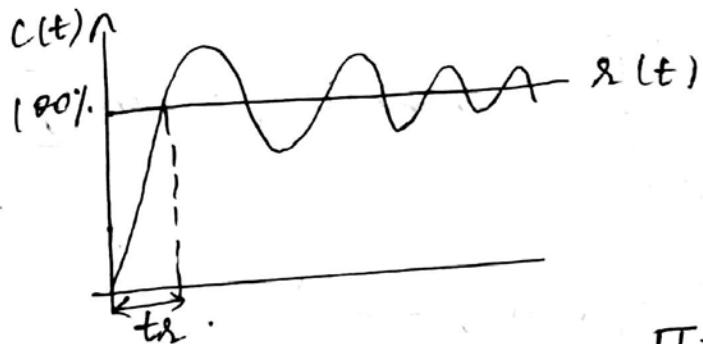
WKT, the output function of II order control

$$C(t) = 1 - e^{-\xi \omega_n t} \left[\sqrt{1-\xi^2} \cos \omega_n t + \xi \sin \omega_n t \right] \quad (i)$$

$$c(t) = 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \cdot \sin(\omega_n t + \phi) \quad (\text{ii})$$

where $\phi = \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi}$.

Rise time (t_r):



The time response of the II-order system to a unit step i/p is given by -

$$c(t) = 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_n t + \phi)$$

where $\phi = \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi}$

at $t = t_r$:

$$c(t_r) = 1 - \frac{e^{-\xi \omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_n t_r + \phi)$$

at $t = t_r$ $c(t) = 100\%$ i.e.

$$1 = 1 - \frac{e^{-\xi \omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_n t_r + \phi)$$

$$\frac{-e^{-\xi \omega_n t_r}}{\sqrt{1-\xi^2}} \cdot \sin(\omega_n t_r + \phi) = 0$$

$$e^{-\xi \omega_n t_r} \sin(\omega_n t_r + \phi) = 0$$

at $t = t_x$ exponent term $\neq 0$.

$$\therefore \sin(\omega_d t_x + \phi) = 0$$

$$\omega_d t_x + \phi = \sin^{-1}(0) = 0, \pi, 2\pi, \dots$$

t_x can't occur at 0.

$$\therefore \omega_d t_x + \phi = \pi$$

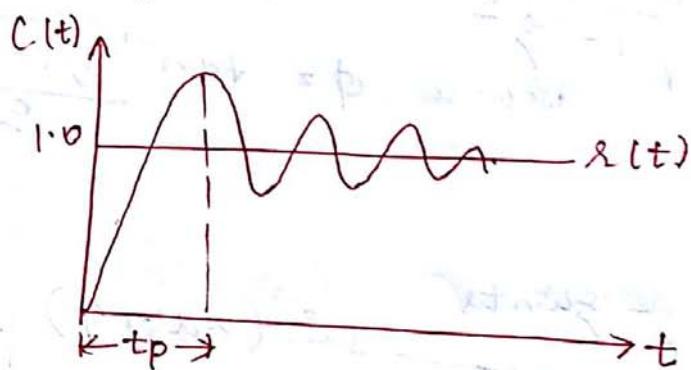
$$\omega_d t_x = \pi - \phi$$

$$t_x = \frac{\pi - \phi}{\omega_d}$$

$$\text{where } \phi = \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi}$$

$$\therefore t_x = \frac{\pi - \left(\tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi} \right)}{\omega_d}$$

Derivation for peak time :-



WKT

$$C(t) \approx 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \cdot \sin(\omega_d t + \phi)$$

$$\text{at } t = t_p$$

$C(t)$ will achieve its maximum, A/C
to maxima theorem

$$\frac{dC(t)}{dt} \Big|_{t=t_p} = 0$$

Differentiate $C(t)$ with respect to t —

$$\frac{dC(t)}{dt} = 0 - \left[\frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} (-\xi \omega_n) \cdot \sin(\omega_d t + \phi); \right. \\ \left. \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \cos(\omega_d t + \phi) \cdot \omega_d \right]$$

$$\frac{dC(t)}{dt} = \left[\frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \xi \omega_n \cdot \sin(\omega_d t + \phi) \right. \\ \left. - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \cos(\omega_d t + \phi) \cdot \omega_d \right]$$

$$\text{at } t = t_p$$

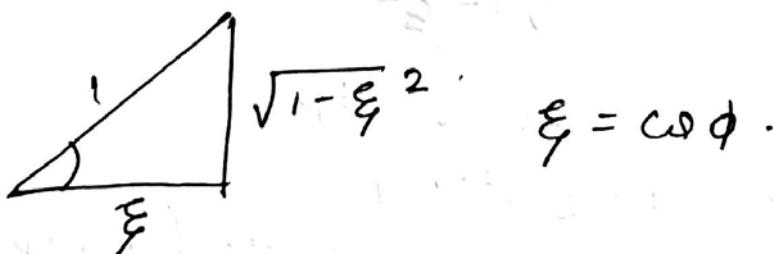
$$\frac{dC(t)}{dt} = \left[\frac{e^{-\xi \omega_n t_p}}{\sqrt{1-\xi^2}} \xi \omega_n \sin(\omega_d t_p + \phi) \right. \\ \left. - \frac{e^{-\xi \omega_n t_p}}{\sqrt{1-\xi^2}} \cos(\omega_d t_p + \phi) \omega_d \right]$$

$$0 = \frac{e^{-\xi \omega_n t_p}}{\sqrt{1-\xi^2}} \left[\xi \omega_n \sin(\omega_d t_p + \phi) - \cos(\omega_d t_p + \phi) \right]$$

$$0 = \xi \omega_n \sin(\omega_d t_p + \phi) - \cos(\omega_d t_p + \phi) \omega_n \sqrt{1-\xi^2}$$

$$0 = \omega_n \left[\xi \sin(\omega_d t_p + \phi) - \cos(\omega_d t_p + \phi) \cdot \sqrt{1-\xi^2} \right]$$

$$0 = \xi \sin(\omega_d t_p + \phi) - \cos(\omega_d t_p + \phi) \sqrt{1-\xi^2}$$



$$0 = \cos \phi \cdot \sin(\omega_d t_p + \phi) - \cos(\omega_d t_p + \phi) \sin \phi$$

$$\sin(\omega_d t_p + \phi - \phi) = 0$$

$$\sin(\omega_d t_p) = 0$$

$$\omega_d t_p \cdot \sin^{-1}(0), \pi, 2\pi, 3\pi \dots$$

t_p can't occur at zero.

$$\omega_d t_p = \pi$$

$$t_p = \frac{\pi}{\omega_d}$$

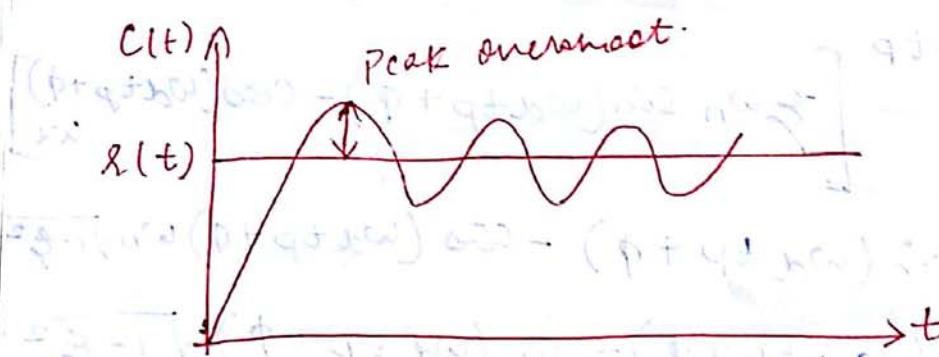
$$t_p = \frac{\pi}{\omega_n \sqrt{1 - \xi^2}}$$

It is the first time for which first overshoot occurs.

$$t_p = \frac{2\pi}{\omega_n \sqrt{1 - \xi^2}} \text{ is the time for}$$

which first undershoot occurs.

Derivation of maximum overshoot (Peak overshoot)



W.K.T

$$C(t) = 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1 - \xi^2}} \sin(\omega_d t + \phi)$$

$$-\xi \omega_n t$$

$$\frac{e}{\sqrt{1 - \xi^2}}$$

$$C(t_p) = 1 - e^{-\xi \omega_n t_p}$$

$$\frac{1}{\sqrt{1 - \xi^2}} \sin(\omega_d t_p + \phi) \quad (i)$$

$$0 = (\phi + \pi + \omega_n t_p)$$

$$0 = (\phi + \pi + \omega_n t_p) \mod \pi$$

$$M_p = C(t_p) - C(\infty)$$

$$M_p = C(t_p) - 1 \quad (\text{ii})$$

$$M_p = \frac{-e^{-\xi \omega_n t_p}}{\sqrt{1-\xi^2}} \sin(\omega_d t_p + \phi). \quad (\text{iii})$$

WKT peak sine, $t_p = \frac{\pi}{\omega_d}$.

$$M_p = \frac{-e^{-\xi \omega_n \frac{\pi}{\omega_d}}}{\sqrt{1-\xi^2}} \cdot \sin\left(\omega_d \frac{\pi}{\omega_d} + \phi\right).$$

$$M_p = \frac{-e^{-\xi \omega_n \frac{\pi}{\omega_d} \sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}} \cdot \sin\left(\omega_d \frac{\pi}{\omega_d} + \phi\right)$$

$$M_p = \frac{e^{-\xi \frac{\pi}{\sqrt{1-\xi^2}}} \cdot -\sin \phi}{\sqrt{1-\xi^2}}$$

$$M_p = \frac{e^{-\xi \frac{\pi}{\sqrt{1-\xi^2}}} \cdot \sin \phi}{\sqrt{1-\xi^2}} \quad \therefore \sin \phi \cdot \frac{1}{\sqrt{1-\xi^2}}$$

$$M_p = \cancel{e^{\cancel{-\xi \frac{\pi}{\sqrt{1-\xi^2}}}} \cdot \cancel{\sqrt{1+\xi^2}} \cdot \cancel{\xi}} \\ \cancel{\sqrt{1-\xi^2}}$$

$$\frac{e^{\cancel{-\xi \frac{\pi}{\sqrt{1-\xi^2}}}} \cdot \sqrt{1-\xi^2}}{\sqrt{1-\xi^2}}$$

$$M_p = e^{\frac{-\xi \pi}{\sqrt{1-\xi^2}}}$$

Approximate exp. for settling time.

Time response for 2nd order S/m to
to unit step input is given by -

since Settling time is measured around
unity. The negative sign in the expression
 $c(t)$ can be neglected.

In the response, exponent term determines
the settling time and not the sine term.
∴ we neglect sine term in the expression
 $c(t)$.

$$c(t) = 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_n t + \phi).$$
$$= \frac{e^{-\xi \omega_n t_s}}{\sqrt{1-\xi^2}} = 0.05 \quad \text{--- (i)}$$

Assume $\xi \ll 1$

$$\sqrt{1-\xi^2} \approx 1$$

∴ eqn (i) becomes -

$$= e^{-\xi \omega_n t_s} = 0.05$$

taking log on both sides

$$-\xi \omega_n t_s \ln 0.05$$

$$+ \xi \omega_n t_s \approx 2.995$$

$$t_s = \frac{2.995}{\xi \omega_n}$$

- 5% tolerance
band.

for 2% tolerance band

$$-\xi \omega_n t_s = \ln(0.02)$$

$$\left[t_s = \frac{3.91}{\xi \omega_n} \right]$$

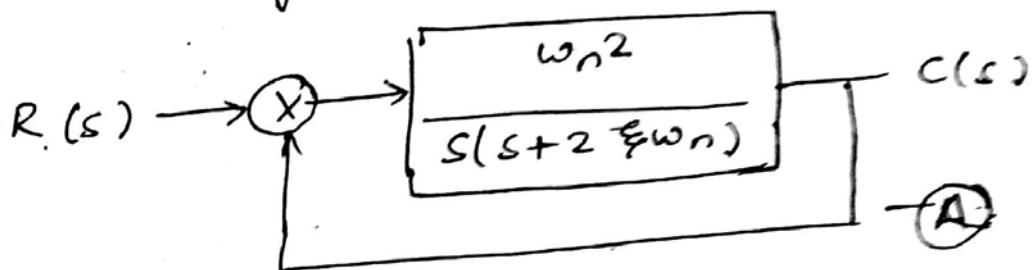
Q The unity

$$G(s)H(s) = \frac{K}{s(s+10)}$$

Determine the system gain K so that system will have

For which value of K , find the rise time, settling time, peak time and peak overshoot. Assume that the system is subjected to the step of IV.

Solⁿ



$$\text{WKT } 1 + G(s)H(s) = 0$$

$$\text{But } G(s)H(s) = \frac{K}{s(s+10)}$$

$$1 + \frac{K}{s(s+10)} = 0$$

$$s(s+10) + K = 0$$

$$s^2 + 10s + K = 0$$

The characteristic equation is found from A -

$$1 + G(s)H(s) = 0$$

$$1 + \frac{\omega_n^2}{s(s+2\xi n)} = 0$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0 \quad \text{--- (ii)}$$

Comparing (i) & (ii)

$$2\xi\omega_n = 10 \quad \text{--- (iii)}$$

$$\omega_n^2 = K \quad \text{--- (iv)}$$

$$\text{Given } \xi = 0.5$$

$$2 \times 0.5 \times \omega_n = 10$$

$$\omega_n = \frac{10}{2 \times 0.5}$$

$$\boxed{\omega_n = 10 \text{ rad/sec}}$$

Substitute the above value in (iv).

$$K = \omega_n^2$$

$$K = 10 \times 10$$

$$\boxed{K = 100}$$

WKT Rise time $t_R = \frac{\pi - \tan^{-1} \left(\frac{\sqrt{1-\xi^2}}{\xi} \right)}{\omega_n \sqrt{1-\xi^2}}$

$$t_R = \frac{\pi - \tan^{-1} \left(\frac{\sqrt{1-(0.5)^2}}{0.5} \right)}{10 \sqrt{1-(0.5)^2}}$$

$$\boxed{t_R = 0.29 \text{ sec}}$$

$$t_P = \frac{\pi}{\omega_n \sqrt{1-\xi^2}} = \frac{\pi}{10 \sqrt{1-(0.5)^2}}$$

$$\boxed{t_P = 0.36 \text{ sec}}$$

$$\text{Settling time, } t \leq \frac{4}{\xi \omega_n} + \frac{4}{0.5 \times 10} = 0.8.$$

$$\text{Peak overshoot} = e^{\frac{-\pi \xi}{\sqrt{1-\xi^2}}}$$

$$= e^{\frac{-\pi 0.5}{\sqrt{1-0.5^2}}}$$

$$= e^{-0.16} = 0.84 = 16\%.$$

~~Measurements conducted on a servo mechanism shows to be suspended to~~

$$\text{be } c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$$

when subjected to step of 1V.

i) Obtain an expression for the closed loop transfer function.

ii) Determine the undamped transfer frequency and damping ratio of the system.

iii) Obtain the unit ramp response of the system.

$$\text{soln: given } c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$$

$$r(t) = u(t) \quad \text{(i)}$$

taking Laplace on both sides for eqn (i) & (ii).

$$C(s) = \frac{1}{s} + \frac{0.2}{s+60} - \frac{1.2}{s+10} \quad \text{--- (iii)}$$

$$R(s) = \frac{1}{s} \quad \text{--- (iv)}$$

Now ~~divide~~ divide (iii) & (iv).

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{s} + \frac{0.2}{s+60} - \frac{1.2}{s+10}}{1/s}$$

$$\frac{C(s)}{R(s)} = \frac{600}{(s+10)(s+60)}$$

$$\frac{C(s)}{R(s)} = \frac{600}{s^2 + 70s + 600}$$

$$= \frac{600}{(s^2 + 70s) \left(1 + \frac{600}{s^2 + 70s}\right)}$$

$$= \frac{\frac{600}{s^2 + 70s} \leftarrow G(s)}{1 + \frac{600}{s^2 + 70s} \times 1 \leftarrow H(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

i) The system characteristic eqⁿ is given by

$$1 + \frac{600}{s^2 + 70s} = 0.$$

$$\frac{s^2 + 70s + 600}{s^2 + 70s} = 0.$$

$$s^2 + 70s + 600 = 0. \quad \text{--- (A)}$$

WKT characteristic eqⁿ is -

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0. \quad \text{--- (B)}$$

Comparing eqⁿ (A) & (B),

$$2\xi\omega_n = 70 \quad \omega_n^2 = 600$$

$$\omega_n = \sqrt{600}$$

$$\xi = \frac{70}{2\omega_n}$$

$$\omega_n = 24.5 \text{ rad/s.}$$

$$\xi = \frac{70}{2 \times 24.5}$$

$$\boxed{\xi = 1.04}$$

$$\frac{C(s)}{R(s)} = \frac{600}{(s+10)(s+60)}$$

$$C(s) = \frac{600}{(s+10)(s+60)} \cdot R(s)$$

For step function, $R(s) = \frac{1}{s^2}$.

$$C(s) = \frac{1}{s^2} \cdot \frac{600}{(s+10)(s+60)}$$

Resolve into partial fraction.

$$C(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s+10)} + \frac{D}{(s+60)}$$

$$600 = A(s^2)(s+10)(s+60) + B(s+10)(s+60) + C(s^2)(s+60) + D(s^2)(s+10).$$

$$A = -0.1167$$

$$B = 1$$

$$C = 0.12$$

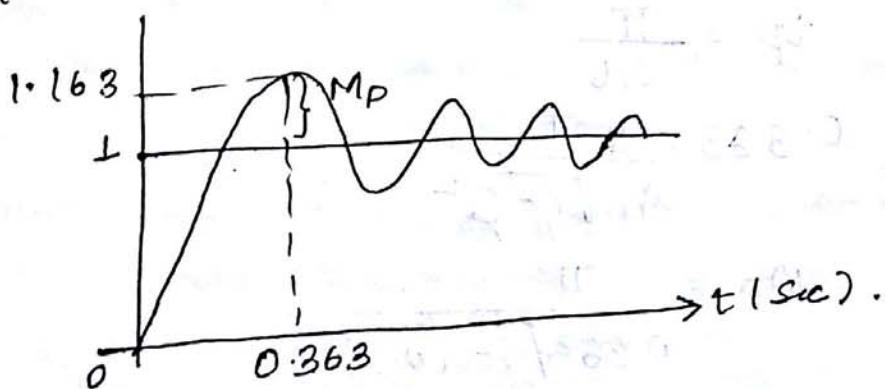
$$D = -0.0003$$

$$C(s) = \frac{-0.1167}{s} + \frac{1}{s^2} + \frac{0.12}{(s+10)} - \frac{0.003}{(s+60)}$$

Taking ILT on both sides.

$$C(t) = -0.1167t + t + 0.12e^{-10t} - 0.003e^{-60t}, t \geq 0.$$

The unit step function is shown below.
Find the transfer function of the photo type system used to model a II-order system.



From the fig we have -
 Peak overshoot, $M_p = 1.163 - 1 = 0.163$

$$M_p = e^{\frac{-\pi \xi}{\sqrt{1-\xi^2}}}$$

$$0.163 = e^{\frac{-\pi \xi}{\sqrt{1-\xi^2}}}$$

$$\ln [0.163] = \frac{-\pi \xi}{\sqrt{1-\xi^2}}$$

$$-1.8140 = -\frac{\pi \xi}{\sqrt{1-\xi^2}}$$

$$1.8140 = (\frac{\pi \xi}{\sqrt{1-\xi^2}})^2 = (-\pi \xi)^2$$

$$3.290 - 0.0225 = \frac{3.290}{0.0225} \xi^2 + \pi^2 \xi^2$$

$$3.290 - 0.0225 = \pi^2 \xi^2 + \frac{3.290}{0.0225} \xi^2$$

$$3.290 - 0.0225 = 4.892 \xi^2 + 13.1602 \xi^2$$

$$\xi^2 = 2.2745 \times 10^{-3}$$

$$\xi, 0.0476$$

$$\xi^2 = 0.2499$$

$$\boxed{\xi = 0.5}$$

From the fig we find that

Peak time, $t_p = 0.3635$

$$\text{WKT } t_p = \frac{\pi}{\omega_n}$$

$$0.363 = \frac{\pi}{\omega_n \sqrt{1-\xi^2}}$$

$$\omega_n = \frac{\pi}{0.363 \sqrt{1-(0.5)^2}}$$

$$\omega_n = 9.99 \text{ rad/sec.}$$

$$\frac{C(s)}{R(s)} = \frac{\omega n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

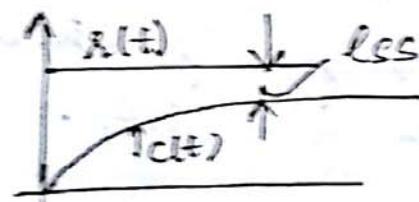
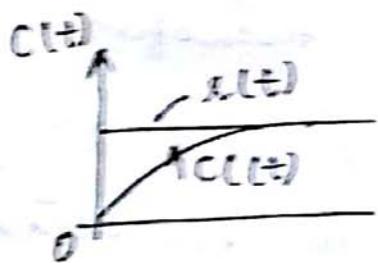
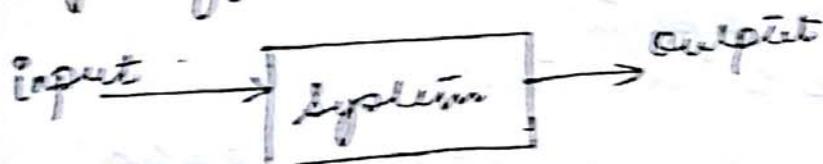
$$\boxed{\frac{C(s)}{R(s)} = \frac{100}{s^2 + 10s + 100}}$$

STEDDY STATE RESPONSE

Output follows the input.

In steady state response the system is checked for its steady state accuracy i.e. whether the output of the system in steady state in following I/P should be capable of an following I/P signal without any steady state error.

The steady state may occur due to drift (Change) in amplifier or bias current in mechanical system. Due to this the desired output and the actual output may different.



The open loop transfer function $G(s)H(s)$ can be written in general form-

$$G(s)H(s) = \frac{k'(s+z_1)(s+z_2) \dots (s+z_n)}{s^n(s+p_1)(s+p_2) \dots (s+p_m+n)}$$

The open loop transfer function $G(s)H(s)$ can be written as -

$$G(s)H(s) = \frac{K^l z_1 \left(1 + \frac{s}{z_1}\right) z_2 \left(1 + \frac{s}{z_2}\right) \cdots z_m \left(1 + \frac{s}{z_m}\right)}{s^n p_1 \left(1 + \frac{s}{p_1}\right) p_2 \left(1 + \frac{s}{p_2}\right) \cdots p_{m+n} \left(1 + \frac{s}{p_{m+n}}\right)}$$

$$\text{Let } K^l = \frac{K^l z_1 z_2 \cdots z_m}{p_1 p_2 \cdots p_{m+n}}$$

The above equation $G(s)H(s)$ is expressed in time constant form $(1+sT)$.

$$\frac{1}{z_1} = Tz_1, \quad \frac{1}{z_2} = Tz_2, \quad \dots$$

$$G(s)H(s) = \frac{K^l (1+Tz_1s)(1+Tz_2s) \cdots (1+Tz_ms)}{s^n (1+Tp_1s)(1+Tp_2s) \cdots (1+Tp_{m+n}s)}$$

In the above expression

K^l = system gain

n = type of the system

If $n=0$, type 0 system

If $n=1$, type 1 system

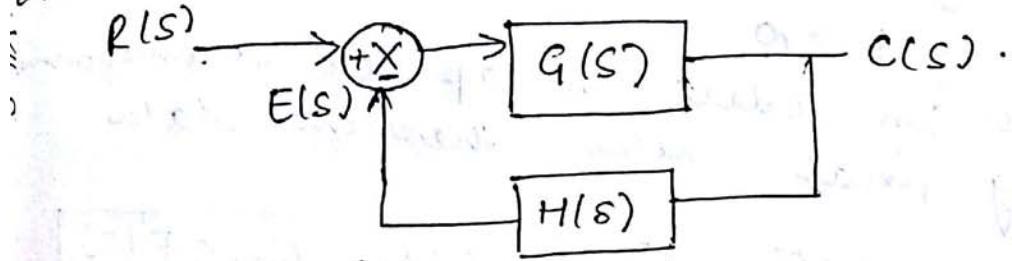
If $n=2$, type 2 system.

Type of m system means no. of poles at the origin of open loop transfer function $G(s)H(s)$.

Higher the type of the system the steady state accuracy of the system increases but the stability decreases since it is difficult to obtain stable state in type 3 or more higher systems. Therefore the practical system are modified to almost type-II.

Steady State error :-

Consider the closed loop system -



The output is given by -

$$C(s) = G(s) E(s).$$

The steady state actuating error signal

$$E(s) = \frac{C(s)}{G(s)} \quad \text{--- (i)}$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}.$$

$$C(s) = \frac{G(s)R(s)}{1 + G(s)R(s)} \quad \text{--- (ii)}$$

Substitute (ii) in (i) —

$$E(s) = \frac{G(s)R(s)}{1 + G(s)H(s)} \cdot \frac{1}{G(s)}$$

$$E(s) = \frac{R(s)}{1 + G(s)H(s)}.$$

∴ The steady state error e_{ss} is given by -

$$e_{ss} = \lim_{s \rightarrow 0} s E(s).$$

$E(s)$ = The error in Laplace domain
i.e. expression in 's'.

In time domain, the corresponding error will be $e(t)$. Now steady state of the system is that state, which remains constant as $t \rightarrow \infty$.

steady state error

$$ESS = \lim_{t \rightarrow \infty} e(t)$$

Now we can relate in Laplace transform by using final value theorem states

that

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

$$\therefore ESS, \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} SE(s)$$

In order to make steady state error in general and independent of system consideration. The output of the system is referred to as position, Rate of change of output as velocity and double rate of change of output as acceleration.

There are 2 types of error constants -

i) Static error constant.

They do not give information regarding the variation of ESS with respect to time.

ii) Dynamic error constant -

ESS in terms of dynamic error coefficient gives information regarding the variation of ESS with respect to time.

Static Error Constant :-

There are three types -

i) static position error constant : K_p

ii) static velocity error constant : K_v

iii) Static acceleration error constant : k_a .

iv) Static position error constant (k_p):

The steady state actuating error signal $E(s)$ to a unit step input is derived by

$$E(s) = \frac{1}{1+G(s)H(s)} \cdot R(s)$$

$R(s) = \frac{1}{s}$ for unit step $1/P$.

$$E(s) = \frac{1}{1+G(s)H(s)} \cdot \frac{1}{s}.$$

The steady state error is given by-

$$e_{ss} = \lim_{s \rightarrow 0} sE(s)$$

$$e_{ss} = \lim_{s \rightarrow 0} s \cdot \frac{1}{1+G(s)H(s)} \cdot \frac{1}{s}$$

$$e_{ss} = \frac{1}{1 + \lim_{s \rightarrow 0} G(s)H(s)}$$

$$e_{ss} = \frac{1}{1 + K_p}$$

$$K_p = \frac{1}{e_{ss}} - 1$$

In general $R(s) = \frac{A}{s}$

$$\therefore K_p = \frac{A}{e_{ss}} - 1$$

$$e_{ss} = \frac{A}{1 + K_p}$$

for type 0 system

$$K_p = \lim_{s \rightarrow 0} G(s)H(s)$$

$$= \lim_{S \rightarrow 0} \frac{K(1+Tz_1S)(1+Tz_2S)\dots}{(1+Tp_1S)(1+Tp_2S)\dots}$$

For type 0 system, $N=0$

$$K_p = K$$

$$ess^2 = \frac{1}{1+K_p}$$

$$\boxed{ess = \frac{1}{1+K}}$$

For type-I system :

$$K_p = \lim_{S \rightarrow 0} G(S) H(S)$$

$$= \lim_{S \rightarrow 0} \frac{K(1+Tz_1S)(1+Tz_2S)\dots}{S^N(1+Tp_1S)(1+Tp_2S)\dots}$$

for type I system, $N=1$.

$$= \lim_{S \rightarrow 0} \frac{K(1+Tz_1S)(1+Tz_2S)\dots}{S^1(1+Tp_1S)(1+Tp_2S)\dots}$$

$$K_p = K \cdot \frac{1}{0 \cdot 1} \cdot \frac{K}{0} = \infty$$

$$ess^2 = \frac{1}{1+K_p} = \frac{1}{1+\infty}$$

$$ess = \frac{1}{\infty} \quad \boxed{ess=0}$$

Type '0' system is capable of responding to a unit step input with a fixed error $\frac{1}{1+K}$.

Type 'I' system & higher sm are capable of responding to a unit step i/p without any ess.

Type 0 is incapable of responding to a unit ramp input.

Type I system can respond with a finite error $\frac{1}{K}$ and type II and higher order systems respond without any steady state error.

∴ Static acceleration error constant (K_a) :-

The steady state error signal $E(s)$ to a unit parabolic input is given by

$$E(s) = \frac{1}{1 + G(s) H(s)} R(s)$$

where $R(s) = \frac{1}{s^3}$ for parabolic i/p.

$$e_{ss} = \lim_{s \rightarrow 0} s E(s)$$

$$= \lim_{s \rightarrow 0} s \cdot \frac{1}{1 + G(s) H(s)} \cdot \left(\frac{1}{s^3} \right)$$

$$= \lim_{s \rightarrow 0} \frac{1}{[1 + G(s) H(s)] s^2}$$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{1}{s^2 + s^2 G(s) H(s)}$$

$$e_{ss} = \frac{1}{\lim_{s \rightarrow 0} s^2 G(s) H(s)}$$

∴ Static error acceleration is defined as -

$$K_a = \lim_{s \rightarrow 0} s^2 G(s) H(s)$$

$$e_{ss} = \frac{1}{K_a}$$

Type '0' system :-

$$G(s) H(s) = \frac{k(1+Tz_1 s)(1+Tz_2 s)}{s^n(1+Tp_1 s)(1+Tp_2 s)} \dots$$

For type 0 $N=0$.

\therefore above eqⁿ is written as -

$$G(s) H(s) = \frac{k(1+Tz_1 s)(1+Tz_2 s)}{(1+Tp_1 s)(1+Tp_2 s)}$$

$$K_a = \lim_{s \rightarrow 0} s^2 G(s) H(s)$$

$$K_a = \lim_{s \rightarrow 0} s^2 \left[\frac{k(1+Tz_1 s)(1+Tz_2 s)}{(1+Tp_1 s)(1+Tp_2 s)} \right]$$

$$\therefore K_a = 0 \quad e_{ss} = \frac{1}{K_a}.$$

$$e_{ss} = \frac{1}{0} \quad \boxed{e_{ss} = \infty}$$

Type '1' system :-

$$G(s) H(s) = \frac{k(1+Tz_1 s)(1+Tz_2 s)}{s^n(1+Tp_1 s)(1+Tp_2 s)} \dots$$

For type 1 $N=1$

$$G(s) H(s) = \frac{k(1+Tz_1 s)(1+Tz_2 s)}{s^1(1+Tp_1 s)(1+Tp_2 s)} \dots$$

$$K_a = \lim_{s \rightarrow 0} \frac{s [k(1+Tz_1 s)(\overrightarrow{1+Tz_2 s})]}{(1+Tp_1 s)(1+Tp_2 s)}$$

$$K_a = \lim_{s \rightarrow 0} s^2 G(s) H(s)$$

$$\lim_{s \rightarrow 0} \frac{s^2 (1+Tz_1 s)(1+Tz_2 s)}{s (1+Tp_1 s)(1+Tp_2 s)}$$

$$\rightarrow K_a = 0 \quad e_{ss} \cdot \frac{1}{K_a} = \frac{1}{0}$$

$e_{ss} = \infty$

Type 2 system -

$$G(s) H(s) = \frac{k(1+Tz_1 s)(1+Tz_2 s)}{s^N (1+Tp_1 s)(1+Tp_2 s)}.$$

$$G(s) H(s) = \frac{k(1+Tz_1 s)(1+Tz_2 s)}{s^2 (1+Tp_1 s)(1+Tp_2 s)}.$$

$$K_a = \lim_{s \rightarrow 0} s^2 G(s) H(s).$$

$$\lim_{s \rightarrow 0} \frac{s^2}{s^2} \left[\frac{k(1+Tz_1 s)(1+Tz_2 s)}{(1+Tp_1 s)(1+Tp_2 s)} \right].$$

$$K_a = \lim_{s \rightarrow 0} \frac{[k(1+Tz_1 s)(1+Tz_2 s)]}{(1+Tp_1 s)(1+Tp_2 s)}$$

$K_a = k$

$$e_{ss} \cdot \frac{1}{K_a} = \frac{1}{k}$$

$k \leftarrow$ finite value

$e_{ss} \cdot \frac{1}{k}$

Type 3 system & higher order system -

$$G(s) H(s) = \frac{k(1+Tz_1 s)(1+Tz_2 s)}{s^N (1+Tp_1 s)(1+Tp_2 s)}$$

$$N = 3$$

$$G(s) H(s) = \frac{k(1+Tz_1 s)(1+Tz_2 s)}{s^3 (1+Tp_1 s)(1+Tp_2 s)}$$

$$K_a = \lim_{s \rightarrow 0} \frac{s^2 [K(1+Tz_1s)(1+Tz_2s)]}{s^3 [(1+Tp_1s)(1+Tp_2s)]}$$

$$\boxed{K_a = \infty}$$

$$ess = \frac{1}{K_a} = \frac{1}{\infty}$$

$$\boxed{ess = 0}$$

Type 0 and Type 1 system are incapable of responding to unit parabolic input.

Type 2 can respond with a finite error $\frac{1}{K_a}$.

Type 3 and higher order systems can respond with no steady state error.

Summary of Steady State errors.

Type of S/m	unit step i/p	unit ramp i/p	unit Parabol i/p
Type 0	$\frac{1}{1+K_p}$	∞	∞
Type 1	0	$\frac{1}{K_V}$	∞
Type 2	0	0	$\frac{1}{K_a}$

For a negative feedback control system having $G(s) = \frac{K}{s^2(s+2)(s+5)}$

Find i) K_p ii) K_V iii) K_a iv) steady state error for unit step, ramp and parabolic i/p

$$K_a = \lim_{s \rightarrow 0} \frac{s^2 [K(1+Tz_1s)(1+Tz_2s)]}{s^3 [(1+Tp_1s)(1+Tp_2s)]}$$

$$K_a = \infty$$

$$ess = \frac{1}{K_a} = \frac{1}{\infty}$$

$$ess = 0$$

Type 0 and Type 1 system are incapable of responding to unit parabolic input.

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Summary of Steady State errors.

Type of S/m	unit step i/p	unit ramp i/p	unit Parabol i/p
Type 0	$\frac{1}{1+K_p}$	∞	∞
Type 1	0	$\frac{1}{K_V}$	∞
Type 2	0	0	$\frac{1}{K_a}$

Q. For a negative feedback control system having $G(s) = \frac{K}{s^2(s+2)(s+5)}$

Find i) K_p ii) K_V iii) K_a iv) steady state error for unit step, ramp and parabolic i/p

$$\text{soln} \quad G(s) = \frac{k}{s^2(s+2)(s+5)}$$

$$K_p = \lim_{s \rightarrow 0} G(s) H(s)$$

$$= \lim_{s \rightarrow 0} \frac{k}{s^2(s+2)(s+5)} \cdot 1$$

$$= \frac{k}{0(0+2)(0+5)}$$

$$\boxed{K_p = \infty}$$

$$K_v = \lim_{s \rightarrow 0} s G(s) H(s)$$

$$= \lim_{s \rightarrow 0} s \cdot \frac{k}{s^2(s+2)(s+5)} \cdot 1$$

$$= \frac{k}{0(0+2)(0+5)}$$

$$\boxed{K_v = \infty}$$

$$K_a = \lim_{s \rightarrow 0} s^2 G(s) H(s)$$

$$= \lim_{s \rightarrow 0} s^2 \cdot \frac{k}{s^2(s+2)(s+5)} \cdot 1$$

$$= \lim_{s \rightarrow 0} \frac{k}{(0+2)(0+5)}$$

$$\boxed{K_a = \frac{k}{10}}$$

Steady state error for unit step :-

$$e_{ss} = \frac{1}{1+K_p} \cdot \frac{1}{1+\infty} = 0.$$

ess for ramp -

$$e_{ss} = \frac{1}{K_v} = \frac{1}{\infty} = 0.$$

Steady state error for parabolic I/P

$$e_{ss} \cdot \frac{1}{K_a} = \frac{1}{K/10} = \frac{10}{K}$$

Q For a unity feedback system

$$G(s) = \frac{K(s+2)}{s(s^3 + 7s^2 + 12s)} \quad \& \quad H(s) = 1$$

Find a) type of system

- b) K_p, K_v, K_a
- c) e_{ss} for unit parabolic I/P.

$$\text{soln: } G(s)H(s) = \frac{K(s+2)}{s(s^3 + 7s^2 + 12s)}$$
$$= \frac{K(s+2)}{s^2(s^2 + 7s + 12)}$$

$$G(s)H(s) = \frac{K}{s^N} \frac{(1+T_{z1}s)(1+T_{z2}s)}{(1+T_{p1}s)(1+T_{p2}s)}$$

On Comparing

$$N=2$$

∴ It is type II - system.

$$K_p = \lim_{s \rightarrow 0} G(s)H(s)$$

$$= \lim_{s \rightarrow 0} \frac{K(s+2)}{s^2(s^2 + 7s + 12)}$$

$$= \frac{1}{0} = \infty$$

$$K_v = \lim_{s \rightarrow 0} sG(s)H(s)$$

$$= \lim_{s \rightarrow 0} \frac{s \cdot K(s+2)}{s^2(s^2 + 7s + 12)}$$

$$= \frac{1}{0} = \infty$$

$$K_a = \lim_{s \rightarrow 0} s^2 G(s) H(s)$$

$$= \lim_{s \rightarrow 0} \frac{s^2 K(s+2)}{s^2(s^2 + 1s + 12)}$$

$$\frac{K(0+2)}{(0+0+12)} = \frac{2K}{12} = \frac{K}{6}$$

Steady state error

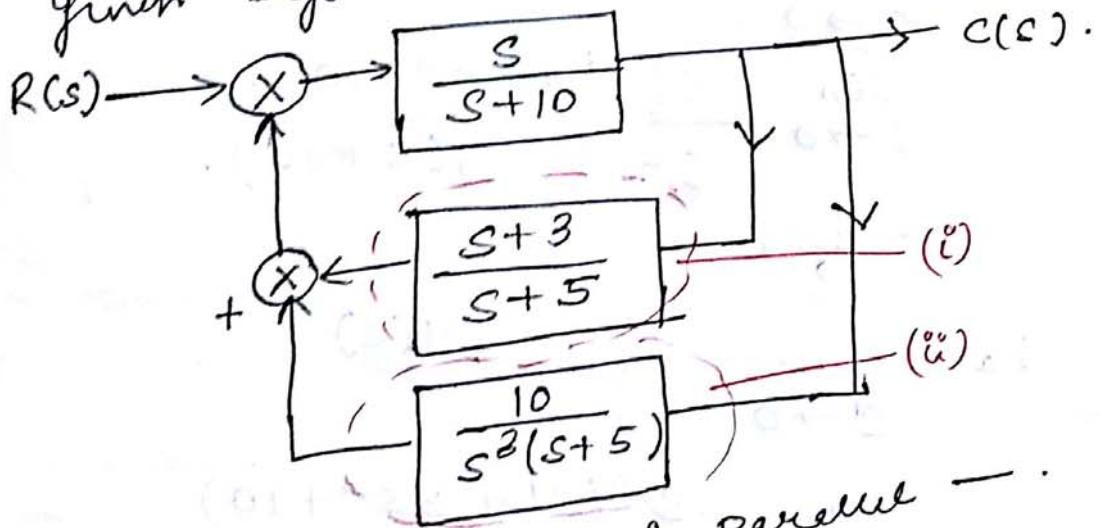
$$ess = \frac{1}{K_a} = \frac{6}{K}$$

^{HW} Q Find KP, KV, KA for unity feedback system whose open loop transfer function is

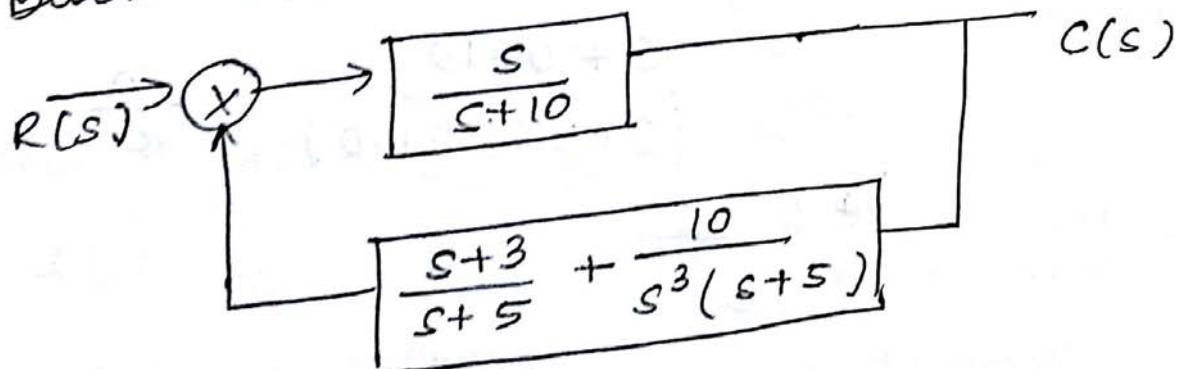
$$G(s) = \frac{100}{s(1+0.1s)(1+0.5s)}$$

$$\begin{aligned} KP &= 0 \\ KV &= 100 \\ KA &= 0 \end{aligned}$$

Q Find the error coefficient for the given system —



Blocks (i) & (ii) are in parallel —



$$\text{From fig, } - \\ G(s) = \frac{s}{s+10} \quad H(s) = \frac{s+3}{s+5} + \frac{10}{s^3(s+5)}$$

$$G(s)H(s) = \frac{s}{s+10} \cdot \frac{(s+3)s^3+10}{s^3(s+5)} \\ = \frac{s^4 + 3s^3 + 10}{s^2(s+5)(s+10)}$$

$$K_P = \lim_{s \rightarrow 0} G(s)H(s)$$

$$= \lim_{s \rightarrow 0} \frac{s^4 + 3s^3 + 10}{s^2(s+5)(s+10)}$$

$$\cancel{(0+0+10)} \quad \frac{1}{0} = \infty$$

$$K_V = \lim_{s \rightarrow 0} sG(s)H(s)$$

$$\lim_{s \rightarrow 0} \frac{s(s^4 + 3s^3 + 10)}{s^2(s+5)(s+10)}$$

$$\frac{1}{0} = \infty$$

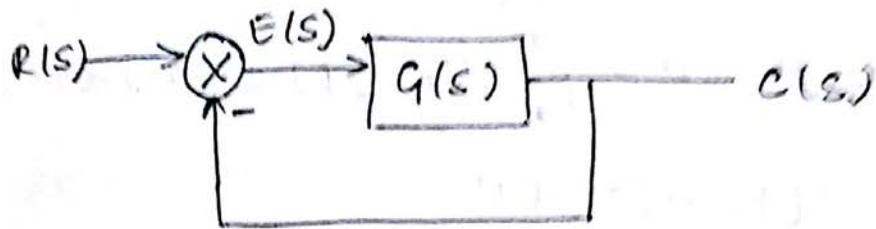
$$K_a = \lim_{s \rightarrow 0} s^2 G(s)H(s)$$

$$\lim_{s \rightarrow 0} \frac{s^2(s^4 + 3s^3 + 10)}{s^2(s+5)(s+10)}$$

$$\frac{0+0+10}{(0+5)(0+10)} = \frac{10}{50}$$

$$K_a = \frac{1}{5}$$

Dynamic error constant or generalized error series.



Consider a negative unity feedback control system as shown above.

From the block diagram, we have -

$$C(s) = E(s) G(s) \quad \text{--- (i)}$$

Closed loop transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} \quad \text{--- (ii)}$$

The error signal is given by -

$$E(s) = R(s) \cdot \frac{1}{1 + G(s)}$$

$$E(s) = R(s) \cdot w(s)$$

where $w(s) = \frac{1}{1 + G(s)}$ is called as error weighting function.

convolution.

$$\mathcal{L}^{-1}[F_1(s) F_2(s)] = \int_{-\infty}^t f_1(t-T) f_2(T) dT$$

$$\mathcal{L}^{-1}[E(s)] = e(t) = \int_0^t \varphi(t-T) w(T) dT \quad \text{--- (iii)}$$

Expand $\varphi(t-T)$ using Taylor's series -

$$\varphi(t-T) = \varphi(t) - T\varphi'(t) + \frac{T^2}{2!} \varphi''(t) + \dots \quad \text{--- (iv)}$$

Sub (iv) in (iii) -

$$e(t) = \int_0^t (\lambda(t) - T\lambda'(t) + \frac{T^2}{2!} \lambda''(t) \dots) \omega(\tau) d\tau$$

$$e(t) = \int_0^t \lambda(t) \omega(\tau) d\tau - \int_0^t T\lambda'(t) \omega(\tau) d\tau + \int_0^t \frac{T^2}{2!} \lambda''(t) \omega(\tau) d\tau.$$

Take limits on both sides as $t \rightarrow \infty$.

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{t \rightarrow \infty} \left[\int_0^t \lambda(t) \omega(\tau) d\tau - \int_0^t T\lambda'(t) \omega(\tau) d\tau + \int_0^t \frac{T^2}{2!} \lambda''(t) \omega(\tau) d\tau \right]$$

Replace $\lambda(t)$ by $\lambda_s(t)$

$\lambda_s(t)$ = steady state of O/P.

$$e_{ss} = \lim_{t \rightarrow \infty} \left[\int_0^t \lambda_s(t) \omega(\tau) d\tau - \int_0^t T\lambda'_s(t) \omega(\tau) d\tau + \int_0^t \frac{T^2}{2!} \lambda''_s(t) \omega(\tau) d\tau \right]$$

$$e_{ss} = \lambda_s(t) \int_0^\infty \omega(\tau) d\tau - \lambda'_s(t) \int_0^\infty T\omega(\tau) d\tau + \frac{\lambda''_s(t)}{2} \int_0^\infty \omega(\tau) d\tau + \dots$$

$$C_0 = (-1)^0 \int_0^\infty \omega(\tau) d\tau.$$

$$C_1 = (-1)^1 \int_0^\infty \omega(\tau) d\tau.$$

$$C_2 = (-1)^2 \int_0^\infty T^2 \omega(\tau) d\tau.$$

$$C_n = (-1)^n \int_0^\infty T^n \omega(\tau) d\tau.$$

$$\text{Ans. } e_{ss} = \varphi_s(t) C_0 + \varphi'_s(t) C_1 + \frac{\varphi''_s(t)}{2} C_2 + \dots$$

This series is known as generalised error series and the coefficient $C_0, C_1, C_2, \dots, C_n$ are known as generalised error coefficient.

Q If $G(s) = \frac{500}{s(s+1)}$ Find C_0, C_1, C_2, e_{ss} for $\varphi(t)$.

SOL WKT

$$e_{ss} = \varphi_s(t) C_0 + \varphi'_s(t) C_1 + \frac{\varphi''_s(t)}{2} C_2 + \dots$$

$$\text{where } C_0 = (-1)^0 \int_0^\infty \omega(\tau) d\tau = \lim_{s \rightarrow 0} \omega(s).$$

$$C_1 = (-1)^1 \int_0^\infty \tau \omega(\tau) d\tau = \lim_{s \rightarrow 0} \frac{d}{ds} [\omega(s)].$$

$$C_2 = (-1)^2 \int_0^\infty \tau^2 \omega(\tau) d\tau = \lim_{s \rightarrow 0} \frac{d^2}{ds^2} [\omega(s)].$$

$$C_0 = \lim_{s \rightarrow 0} \omega(s)$$

$$= \lim_{s \rightarrow 0} \frac{1}{1+G(s)} - \frac{G(s+1)}{G'(s+1) + 500}$$

$$\boxed{C_0 = 0}$$

$$C_1 = \lim_{s \rightarrow 0} \frac{d}{ds} \left[\frac{s^2+s}{s^2+s+500} \right]$$

$$= \lim_{s \rightarrow 0} \frac{(s^2+s+500)(2s+1) - (s^2+s)[2s+1]}{(s^2+s+500)^2}$$

$$= \frac{2sH[\varphi^2 + \varphi + 500 - \varphi^2 - \varphi]}{(s^2+s+500)^2}$$

$$\frac{(s^2+1)500}{(s^2+s+500)^2} = \frac{1}{500} = 2 \text{ m.}$$

$$C_2 = \lim_{s \rightarrow 0} \frac{d}{ds} \left[\frac{500(2s+1)}{(s^2+s+500)^2} \right]$$

$$= \left[\frac{(s^2+s+500)^2 1000 - 500(2s+1) 2(s^2+s+500)}{(s^2+s+500)^4} \right] \frac{2s+1}{2s+1}$$

$$= \lim_{s \rightarrow 0} \left[\frac{(s^2+s+500) 1000 - 1000(2s+1)^2}{(s^2+s+500)^3} \right]$$

$$= \frac{500000 - 1000}{(500)^3} = 0.3992 \approx 4 \text{ m.}$$

$$ess = 0 + \frac{1}{500} \dot{e}_s(t) + \frac{1}{2 \cdot 50 \times 2} \ddot{e}_s(t).$$

Q The unity feedback system has $G(s) = \frac{1}{s(1+2s)}$. The input to the system is described by $e(t) = 2 + 4t + 6t^2 + 2t^3$. Determine the generalised error coefficient and express ess as function of time.

Soln. $C_0 = \lim_{s \rightarrow 0} \frac{1}{1 + \frac{1}{s(1+2s)}} = \frac{s+2s^2}{s+2s^2+1}$.

$\boxed{C_0 = 0.}$

$$C_1 = \lim_{s \rightarrow 0} \frac{d}{ds} \left[\frac{s+2s^2}{s+2s^2+1} \right]$$

$$= \frac{(s+2s^2+1)(1+4s) - (s+2s^2)[1+4s]}{(s+2s^2+1)^2}$$

$$C_1 = \lim_{s \rightarrow 0} \frac{(1+4s)[s+2s^2+1 - s^2 - 2s^2]}{(s+2s^2+1)^2}$$

$$\boxed{C_1 = \frac{1}{1} = 1}$$

$$C_2 = \lim_{s \rightarrow 0} \frac{d}{ds} \left[\frac{1+4s}{(s+2s^2+1)^2} \right]$$

$$= \lim_{s \rightarrow 0} \frac{(s+2s^2+1)^4 - (1+4s)(1+4s)}{2(s+2s^2+1)} \cdot$$

$$\frac{(2s^2+s+1)^4}{(2s^2+s+1)^4}$$

$$= \lim_{s \rightarrow 0} \left[\frac{(s+2s^2+1)^4 - 2(1+4s)^2}{(s+2s^2+1)^3} \right]$$

$$\therefore \frac{4-2}{1} = 2 \quad \boxed{C_2 = 2}$$

$$e_{ss} = \underline{\underline{\mathcal{L}S'(t)}} + 2\underline{\underline{\mathcal{L}S''(t)}}$$