

Thus

$$Tds = dh - v dP$$

The second  $Tds$  equation is extensively used in the study of compressible flow machines. In terms of stagnation properties

$$Tds = dh_0 - v_0 dP_0$$

For an incompressible fluid undergoing an isentropic process (i.e.  $ds = 0$ ) as in fans, the ideal change in stagnation enthalpy is

$$(\Delta h_0)_s = v_0 \Delta P_0 = \Delta P_0 / \rho_0 = \Delta P_0 / \rho$$

Since  $v_0 = 1/\rho_0$  and  $\rho_0 = \rho$

## EFFICIENCIES OF COMPRESSORS

Fig. 1.2 shows the reversible and irreversible adiabatic compression processes on the enthalpy-entropy diagram. The initial condition of the fluid is represented by state-1. The stagnation point corresponding to this state is 01. The final condition of the fluid is denoted by state-2 and the corresponding stagnation point is 02. If the process were reversible, the final fluid static and stagnation conditions would be  $2^*$  and  $02^*$  respectively.

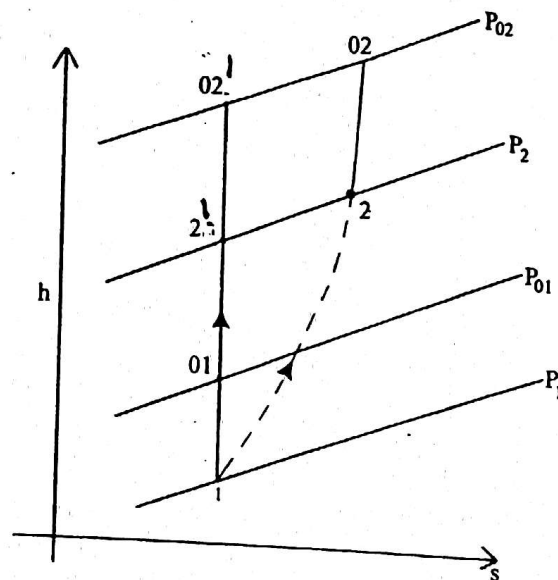


Figure 1.2 Reversible and irreversible compression processes

Process 1-2 is the actual compression process and is accompanied by an increase in entropy. Process 1-2\* is the ideal compression process. The efficiencies of compressors may be defined in terms of either stagnation or static properties of the fluid or even a combination of both. The following are the commonly used compressor efficiencies:

### 1. Total-to-Total Efficiency

It is an efficiency based on stagnation properties at entry and exit.

$$\eta_{t-t} = \frac{W_{ideal}}{W_{actual}} = \frac{h_{02}^1 - h_{01}}{h_{02} - h_{01}}$$

### 2. Static-to-Static Efficiency

It is an efficiency based on static properties at entry and exit.

$$\eta_{s-s} = \frac{W_{ideal}}{W_{actual}} = \frac{h_2^1 - h_1}{h_2 - h_1}$$

### 3. Polytropic Efficiency

A compressor stage can be viewed as made up of an infinite number of small stages. To account for a compression in an infinitesimal stage, polytropic efficiency is defined for an elemental compression process. Consider a small compressor stage as shown in Fig. 1.3 between pressures  $P$  and  $P+dP$ .

The polytropic efficiency of a compressor stage is defined as

$$\eta_p = \frac{dT_s}{dT} \quad (1.1)$$

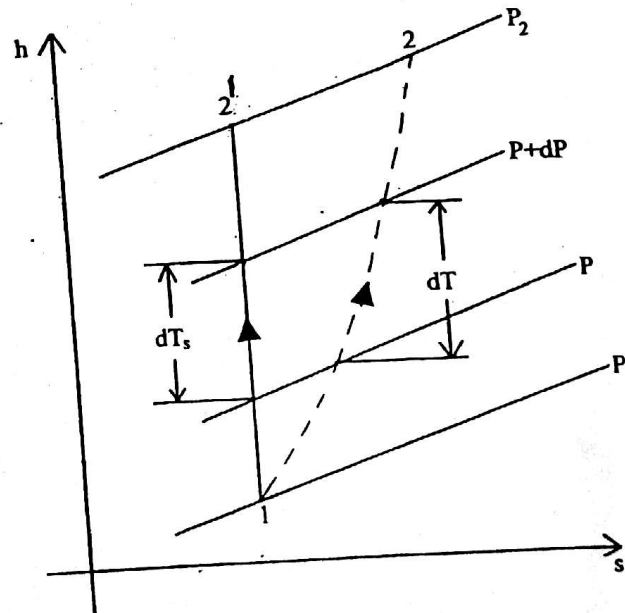


Figure 1.3 Compression process in infinitesimal and finite compressor stages

For an isentropic process, the relationship between pressure and temperature is given by

$$\frac{T}{P^{(\gamma-1)/\gamma}} = \text{constant}$$

(1.2)

Differentiating equation (1.2) and substituting equation (1.1), we get

$$dT \cdot \eta_p = \left[ \frac{r-1}{r} \times P^{-\left(\frac{1}{r}\right)} \right] dp \times \text{constant}$$

Constant value is obtained from equation (1.2).

Therefore,

$$\frac{dT}{T} \cdot \eta_p = \left( \frac{r-1}{r} \right) \cdot \frac{dp}{P} \quad (1.3)$$

Integrating between the limits of the full compression from  $P_1$  to  $P_2$  we get

$$\ln \left( \frac{T_2}{T_1} \right) = \frac{r-1}{r} \cdot \frac{1}{\eta_p} \cdot \ln \left( \frac{P_2}{P_1} \right)$$

Rearranging,

$$\frac{T_2}{T_1} = \left( \frac{P_2}{P_1} \right)^{\frac{r-1}{\eta_p r}} \quad (1.4)$$

If the irreversible adiabatic compression process is assumed to be equivalent to a polytropic process with polytropic index,  $n$ , the following relationship between temperature and pressure will exist.

$$\left( \frac{T_2}{T_1} \right) = \left( \frac{P_2}{P_1} \right)^{\frac{n-1}{n}} \quad (1.5)$$

Equating eqns (1.4) and (1.5),

$$\left( \frac{P_2}{P_1} \right)^{\frac{r-1}{\eta_p r}} = \left( \frac{P_2}{P_1} \right)^{\frac{n-1}{n}}$$

Comparing the power,

$$\frac{r-1}{\eta_p \cdot r} = \frac{n-1}{n}$$

or

$$\eta_p = \frac{r-1}{r} \cdot \frac{n}{n-1}$$

[The polytropic efficiency is also called as small stage or infinitesimal stage efficiency]

A typical value of polytropic efficiency for a compressor is 0.88 and in the initial design calculation it is often assumed that  $\eta_p = \eta_s$ , where  $\eta_s$  is the stage efficiency.

Alternatively, polytropic index of compression in the actual process is

$$n = \frac{r \eta_p}{1 - r(1 - \eta_p)}$$

#### 4. Finite Stage Efficiency

A stage with a finite pressure drop is a finite stage. Taking static values of temperature and pressure (Fig. 1.2) and assuming perfect gas, stage efficiency is defined as

$$\eta_s = \frac{T_{2s} - T_1}{T_2 - T_1}$$

The finite stage or stage efficiency can be expressed in terms of the small stage or polytropic efficiency,

$$T_{2s} - T_1 = T_1 \left( \frac{T_{2s}}{T_1} - 1 \right) = T_1 \left( \left( \frac{P_2}{P_1} \right)^{\frac{\gamma-1}{\gamma}} - 1 \right)$$

and

$$T_2 - T_1 = T_1 \left( \frac{T_2}{T_1} - 1 \right) = T_1 \left( \left( \frac{P_2}{P_1} \right)^{\frac{\gamma-1}{\gamma \eta_p}} - 1 \right)$$

Therefore,

$$\eta_s = \frac{\left( \frac{P_2}{P_1} \right)^{\frac{\gamma-1}{\gamma}} - 1}{\left( \frac{P_2}{P_1} \right)^{\frac{\gamma-1}{\gamma \eta_p}} - 1}$$

For a multistage compressor with a constant stage pressure ratio, the overall pressure ratio is given by

$$\frac{P_{N+1}}{P_1} = \left( \frac{P_2}{P_1} \right)^N$$

where  $N$  is the number of stages and  $P_{N+1}$  is the pressure at the end of the  $N^{th}$  stage. Therefore, the overall efficiency is

$$\eta_c = \frac{\left( \frac{P_2}{P_1} \right)^{N \left( \frac{\gamma-1}{\gamma} \right)} - 1}{\left( \frac{P_2}{P_1} \right)^{N \left( \frac{\gamma-1}{\gamma \eta_p} \right)} - 1}$$

The overall efficiency in terms of overall pressure ratio is

$$\eta_c = \frac{\left( \frac{P_{N+1}}{P_1} \right)^{\frac{\gamma-1}{\gamma}} - 1}{\left( \frac{P_{N+1}}{P_1} \right)^{\frac{\gamma-1}{\gamma \eta_p}} - 1}$$

## PREHEAT FACTOR IN COMPRESSORS

Consider a two stage compressor working between  $P_{01}$  and  $P_{03}$  as shown in Fig. 1.4. In isentropic flow, the outlet conditions of the gas for the first stage and second stage are at  $O2^1$  and  $O3^1$  respectively, whereas the actual outlet

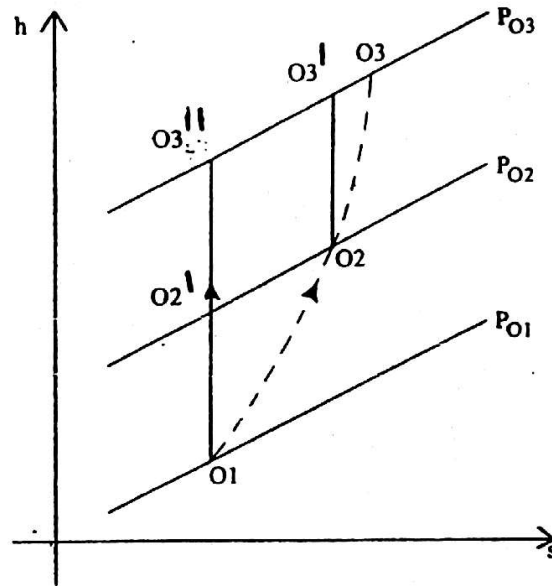


Figure 1.4 Preheat effect in a multistage compressor

conditions are at  $O2$  and  $O3$ . Corresponding to first stage and second stage respectively. The subscript 's' refers to constant entropy and '0' refers to total conditions of the fluid. If the stage efficiencies were the same, the total actual work input to the different individual stages would be

$$W = \frac{1}{\eta_s} [W_{s1} + W_{s2}] = \frac{1}{\eta_s} [(h_{O2^1} - h_{O1}) + (h_{O3^1} - h_{O2})]$$

$$W = \frac{1}{\eta_s} \sum_{i=1}^2 W_{si} \quad (1.6)$$

where  $\sum_{i=1}^2 W_{si}$  is the isentropic work input to the two stage compressor and is the sum of the stage isentropic works. For a compressor with 'n' stages,

$$W = \frac{1}{\eta_s} \sum_{i=1}^N W_{si} \quad (1.7)$$

Consider now, a single stage compressor raising the fluid pressure from  $P_{01}$  to  $P_{03}$ . The actual work input that would be supplied is

$$W = \frac{1}{\eta_c} W_s \quad (1.7a)$$

where  $\eta_c$  is the overall compressor efficiency and  $W_s$  is the isentropic work.

$$W_s = h_{03}^{11} - h_{01}$$

The actual work input is the same for both single stage and multistage compression processes.

$$W = h_{03} - h_{01}$$

Then from equ's 1.7 and 1.7a,

$$\frac{\eta_s}{\eta_c} = \frac{\sum_{i=1}^N W_{si}}{W_s} \quad (1.8)$$

Since the constant pressure lines diverge in the direction of increasing entropy on h-s diagram, the isentropic enthalpy rise across each stage increases even for a constant stagnation pressure rise  $\Delta P_0$  across each stage. Then, the sum of the stage isentropic enthalpy rises is greater than the isentropic enthalpy rise in a single stage compression.

For a two stage compressor

$$(h_{02}^1 - h_{01}) + (h_{03}^1 - h_{02}) > (h_{03}^{11} - h_{01})$$

i.e.  $\sum_{i=1}^2 W_{si} > W_s$

For N stages,

$$\frac{\sum_{i=1}^N W_{si}}{W_s} > 1$$

Equation (1.8) can be written as

$$\frac{\eta_c}{\eta_s} = \frac{W_s}{\sum_{i=1}^N W_{si}} = P.F. \quad (1.8a)$$

That is, the ratio of  $W_s$  to  $\sum_{i=1}^N W_{si}$  is known as the Preheat factor (P.F)

$$P.F. = \frac{W_s}{\sum_{i=1}^N W_{si}} < 1$$

The preheat factor is less than unity. Then, equation (1.8a) becomes

$$\frac{\eta_c}{\eta_s} < 1$$

or

$$\eta_c < \eta_s$$

i.e., the overall compressor efficiency  $\eta_c$  is less than the compressor stage efficiencies  $\eta_s$ .

Consider again Fig. (1.4) for a first stage compression, state  $O2$  may be obtained after an ideal compression from  $O1$  to  $O2'$  followed by "preheatin" of the fluid from state  $O2'$  to  $O2$  at constant pressure ( $T_{O2} > T_{O2'}$ ).

This inherent thermodynamic effect that reduces the efficiency of a multistage compressor is called the *preheat effect*.

## EFFICIENCIES OF TURBINES

The enthalpy-entropy diagram for flow both reversible and irreversible through a turbine is shown in Fig. 1.5. The static condition of the fluid at inlet is determined by state 1, with state  $O1$ , as the corresponding stagnation state. The final static properties are determined by the state 2, with  $O2$ , as the

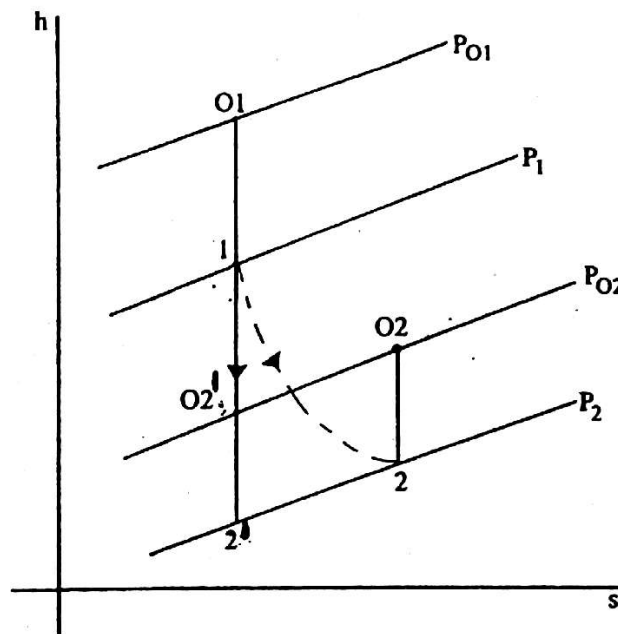


Figure 1.5 Reversible and irreversible expansion processes

corresponding stagnation state. If the process were reversible, the final fluid static state would be  $2'$  and the stagnation state would be  $O2'$ .

Process  $1 - 2$  is the actual expansion process and process  $1 - 2'$  is the isentropic or ideal expansion process. In turbines, the efficiencies may be defined using either the static or the stagnation properties of the fluid or even a combination of both. The commonly used turbine efficiencies are

### 1. Total-to-Total Efficiency

It is an efficiency based on stagnation properties at inlet and outlet.

$$\eta_{t-t} = \frac{W_{actual}}{W_{ideal}}$$

$$\eta_{t-t} = \frac{h_{O1} - h_{O2}}{h_{O1} - h_{O2'}}$$

## 2. Total-to-Static Efficiency

It is an efficiency in which the ideal work is based on stagnation property at inlet and static property at outlet.

$$W_{ideal} = h_{01} - h_{2'}^I$$

$$\eta_{t-s} = \frac{W_{actual}}{W_{ideal}} = \frac{h_{01} - h_{02}}{h_{01} - h_{2'}^I}$$

## 3. Polytropic Efficiency

A turbine stage can be considered as made up of an infinite number of small or infinitesimal stages. Then to account for expansion in an infinitesimal turbine

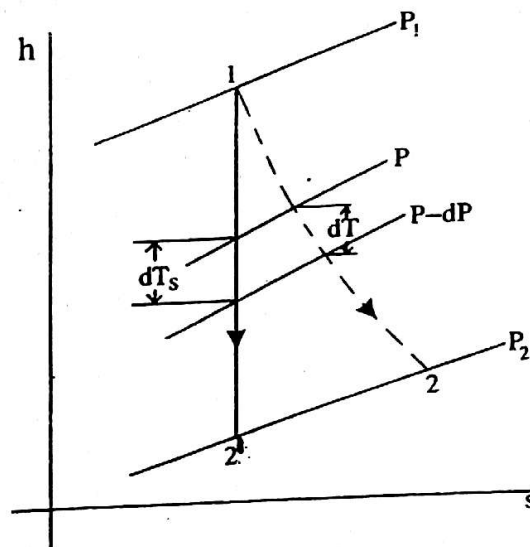


Figure 1.6 Expansion process in infinitesimal and finite turbine stages

stage, a small stage or infinitesimal stage or polytropic efficiency is defined. Consider a small stage (Fig. 1.6) between pressures  $P$  and  $P - dp$ . The efficiency of this turbine stage is defined as

$$\eta_p = \frac{dT}{dT_s} \quad (1.9)$$

For an isentropic process

$$\frac{T}{P^{(r-1)/r}} = \text{Constant} \quad (1.10)$$

Differentiating eqn. (1.10), we get

$$dT_s = \text{Constant} \left[ P^{-(1/r)} \cdot \frac{(r-1)}{r} \right] dp$$

But  $dT_s = \frac{dT}{\eta_p}$  and constant  $= \frac{T}{P^{(r-1)/r}}$

then

$$\frac{dT}{\eta_p} = \frac{T}{P^{(r-1)/r}} \left[ P^{-(1/r)} \cdot \frac{(r-1)}{r} \right] dp$$

$$\frac{dT}{T} = \eta_p \cdot \frac{r-1}{r} \cdot \frac{dP}{P} \quad (1.11)$$

Integrating between the limits of the overall expansion between  $P_1$  and  $P_2$

$$\ln\left(\frac{T_2}{T_1}\right) = \eta_p \cdot \frac{r-1}{r} \cdot \ln\left(\frac{P_2}{P_1}\right)$$

Rearranging,

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\eta_p \left(\frac{r-1}{r}\right)} \quad (1.12)$$

Assuming the irreversible adiabatic expansion (1-2) as equivalent to a polytropic process with index  $n$ , the temperature and pressure are related by

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} \quad (1.13)$$

Equating eqns. (1.12) and (1.13),

$$\left(\frac{P_2}{P_1}\right)^{\eta_p \left(\frac{r-1}{r}\right)} = \left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}}$$

Comparing the powers,

$$\eta_p \left(\frac{r-1}{r}\right) = \frac{n-1}{n}$$

or

$$\eta_p = \frac{r}{(r-1)} \cdot \frac{n-1}{n}$$

Alternatively, the index of expansion in the actual process is expressed as

$$n = \frac{r}{r - (r-1)\eta_p}$$

When  $\eta_p = 1$ ,  $n = r$ . The actual expansion of process curve (1-2) coincides with the isentropic expansion line (1-2').

#### 4. Finite Stage Efficiency

The stage efficiency, considering static value of temperature and pressure (Fig. 1.6.), is defined as

$$\eta_s = \frac{T_1 - T_2}{T_1 - T_2^i}$$

The stage efficiency can now be expressed in terms of polytropic efficiency

$$T_1 - T_2^i = T_1 \left( 1 - \left( \frac{P_2}{P_1} \right)^{\frac{r-1}{r}} \right)$$

and

$$T_1 - T_2 = T_1 \left( 1 - \frac{T_2}{T_1} \right) = T_1 \left( 1 - \left( \frac{P_2}{P_1} \right)^{\frac{\eta_p(r-1)}{r}} \right)$$

Therefore,

$$\eta_s = \frac{1 - \left( \frac{P_2}{P_1} \right)^{\frac{\eta_p(r-1)}{r}}}{1 - \left( \frac{P_2}{P_1} \right)^{\frac{r-1}{r}}}$$

The same equation can be used to determine the overall efficiency of a multistage turbine, except that the stage pressure ratio is replaced by the overall pressure ratio. The overall efficiency, for a  $N$ -stage turbine with a constant stage pressure ratio, can be expressed as

$$\frac{P_{N+1}}{P_1} = \left( \frac{P_2}{P_1} \right)^N$$

$$\eta_t = \frac{1 - \left( \frac{P_2}{P_1} \right)^{N \left( \frac{\eta_p(r-1)}{r} \right)}}{1 - \left( \frac{P_2}{P_1} \right)^{N \left( \frac{r-1}{r} \right)}}$$

#### REHEAT FACTOR IN TURBINES

Consider a turbine with two stages where the fluid (perfect gas) expands from  $P_{01}$  to  $P_{03}$  as shown on the  $h-s$  diagram, Fig. 1.7. State-01 is the initial condition at the entry of the first stage and 02s is the condition that would be reached at the first stage exit if the expansion process had been isentropic. The actual expansion leads

to a final state-02 which has higher entropy than that of state 02'. The corresponding exit conditions for the second stage are 03' and 03 respectively.

The isentropic work done by the two stage turbine is the sum of the stage isentropic works.

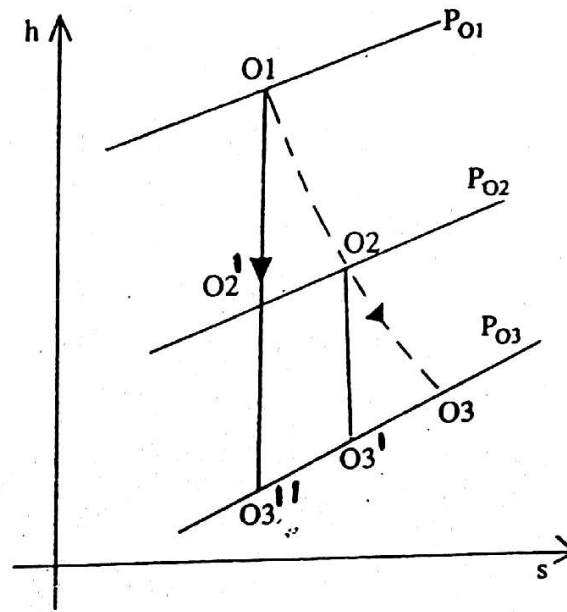


Figure 1.7 Reheat effect in a multistage turbine

$$\begin{aligned} \sum_{i=1}^2 W_{si} &= W_{s1} + W_{s2} \\ &= (h_{01} - h_{02}') + (h_{02} - h_{03}') \end{aligned} \quad (1.14)$$

Consider now a turbine in which the fluid expands from  $P_{01}$  to  $P_{03}$  in one stage. The isentropic work is

$$W_s = h_{01} - h_{03}' \quad (1.15)$$

In both the cases, the actual work done is

$$W = h_{01} - h_{03}$$

The constant pressure line diverges in the  $h - s$  diagram as the entropy increases. Therefore, the isentropic enthalpy drop across a stage increases for a constant stagnation pressure drop  $\Delta P_0$  across each stage. Consequently, the sum of the stage isentropic enthalpy drops is greater than the isentropic enthalpy drop in a single stage expansion. That is

$$(h_{01} - h_{02}') + (h_{02} - h_{03}') > (h_{01} - h_{03}')$$

or

$$\sum_{i=1}^2 W_{si} > W_s$$

For a turbine with  $N$ -stages, we have

$$\frac{\sum_{i=1}^N W_{s_i}}{W_s} > 1 \quad (1.16)$$

The ratio of  $\sum_{i=1}^N W_{s_i}$  to  $W_s$  is called the reheat factor (R.F).

$$\boxed{\text{R.F} = \frac{\sum_{i=1}^N W_{s_i}}{W_s} > 1} \quad (1.17)$$

The magnitude of the reheat factor in multistage turbines is about 1.03 or 1.04.

If the stage efficiencies were the same, the total actual work output from the various individual stages would be

$$W = \eta_s \sum_{i=1}^N W_{s_i} \quad (1.18)$$

The actual work output that would be obtained from a single stage expansion is

$$W = \eta_t W_s \quad (1.19)$$

where  $\eta_t$  is the overall turbine efficiency. Combining equations (1.18) and (1.19), we get

$$\frac{\eta_t}{\eta_s} = \frac{\sum_{i=1}^N W_{s_i}}{W_s} \quad (1.20)$$

From eqn. (1.16), we find that

$$\eta_t > \eta_s$$

That is, the overall turbine efficiency  $\eta_t$  is greater than the turbine stage efficiencies  $\eta_s$ .

Combining eqns. (1.17) and (1.20), we have the following relation.

$$\boxed{\text{R.F} = \eta_t / \eta_s} \quad (1.21)$$

Consider again, Fig.(1.7) for a first stage expansion. It is seen that the final state -02 may be obtained after an ideal expansion from 01 - 02' followed by a 'reheating' of the fluid from state 02' to 02 at constant pressure ( $T_{0_2} > T_{0_2'}$ )

Hence, the fluid at 02 has a greater availability than the fluid at 02' ( $h_{0_2} > h_{0_2'}$ ). An expansion from state 02 to a lower pressure must necessarily result in a larger output than that obtainable from state 02'. This effect that makes  $\text{RF} > 1$  is called the *reheat effect*.

The reheat factor for the expansion of a perfect gas in an  $N$ -stage turbine, assuming that the stage efficiencies  $\eta_s$  and the pressure ratios  $P_{0i}/P_{0i+1}$ , where  $i = 1, 2, \dots, n$  for all the stages are equal, is expressed in terms of stage pressure ratio as follows. For the first stage,

$$T_{02}^1 = T_{01} (P_{02}/P_{01})^{\frac{\gamma-1}{\gamma}}$$

so that

$$T_{01} - T_{02}^1 = T_{01} \left[ 1 - (P_{02}/P_{01})^{\frac{\gamma-1}{\gamma}} \right] \quad (1.22)$$

Let

$$\mu = 1 - (P_{02}/P_{01})^{\frac{\gamma-1}{\gamma}}$$

Then

$$T_{01} - T_{02}^1 = T_{01} \mu$$

and

$$T_{01} - T_{02} = \eta_s T_{01} \mu$$

or

$$\boxed{T_{02} = T_{01} (1 - \mu \eta_s)} \quad (1.23)$$

For the second stage

$$T_{02} - T_{03}^1 = T_{02} \left[ 1 - (P_{03}/P_{02})^{\frac{\gamma-1}{\gamma}} \right]$$

but

$$\frac{P_{03}}{P_{02}} = \frac{P_{02}}{P_{01}}$$

Therefore

$$T_{02} - T_{03}^1 = T_{02} \mu$$

and

$$T_{02} - T_{03} = \eta_s T_{02} \mu$$

$$T_{03} = T_{02} (1 - \eta_s \mu) = T_{01} (1 - \mu \eta_s)^2 \quad (1.24)$$

For the  $N$ th stage

$$\boxed{T_{0N+1} = T_{01} (1 - \mu \eta_s)^{N-1}}$$

The actual work output is

$$\begin{aligned} W &= C_P \eta_s \mu [T_{01} + T_{02} + \dots + T_{0(N+1)}] \\ &= C_P \eta_s \mu T_{01} [1 + (1 - \mu \eta_s) + (1 - \mu \eta_s)^2 + \dots + (1 - \mu \eta_s)^{N-1}] \end{aligned}$$

The terms within the brackets are of the form  $1 + r + r^2 + \dots + r^{n-1}$  which is a geometric series with common ratio  $r$ . The solution is  $\frac{1 - r^n}{1 - r}$ .

In this case,  $r = (1 - \mu \eta_s)$ . Then, the equation reduces to

$$\begin{aligned} &= C_P T_{01} [1 - (1 - \mu \eta_s)^N] \\ &= \eta_s \sum_{i=1}^N W_{si} \end{aligned} \quad (1.25)$$

The isentropic work output from a single stage expansion from  $P_{01}$  to  $P_{0(N+1)}$  may be obtained from eqn. (1.25) by setting  $\eta_s = 1$ .

$$\begin{aligned} W_s &= C_P T_{01} [1 - (1 - \mu)^N] \\ &= C_P T_{01} \left[ 1 - R_0^{\frac{r-1}{r}} \right] \end{aligned} \quad (1.26)$$

where,  $R_0 = P_{0(N+1)} / P_{01}$  represents the overall pressure ratio. Since

$$\begin{aligned} \text{R.F} &= \frac{\sum_{i=1}^N W_{si}}{W_s} \\ &= \frac{C_P T_{01}}{\eta_s} [1 - (1 - \mu \eta_s)^N] / C_P T_{01} \left[ 1 - R_0^{\frac{r-1}{r}} \right] \\ \boxed{\text{R.F} &= \frac{[1 - (1 - \mu \eta_s)^N]}{\eta_s [1 - R_0^{r-1/r}]}} \end{aligned}$$

## SOLVED PROBLEMS

**Example 1.1** The initial and final total pressures of a fluid are 1 bar and 10 bar respectively. The initial total temperature is 10°C. What is the work of compression for adiabatic steady flow with a total-to-total efficiency of 75% if (a) the fluid is liquid water and (b) the fluid is air as a perfect gas.

### Solution

$$P_{01} = 1 \text{ bar} \quad P_{02} = 10 \text{ bar} \quad T_{01} = 283 \text{ K.} \quad \eta_{tt} = 0.75$$

(a) If the fluid is liquid water

Since the fluid is incompressible  $v_0 = \text{constant} = \frac{1}{\rho}$ ,  $\rho = \text{density of water}$ .