Module 5

Game Theory

1. Introduction to Game Theory

Game theory is a type of decision theory in which one's choice of action is determined after taking into account all possible alternatives available to an opponent playing the same game, rather than just by the possibilities of several outcome results. Game theory does not insist on how a game should be played but tells the procedure and principles by which action should be selected. Thus it is a decision theory useful in competitive situations.

Game is defined as an activity between two or more persons according to a set of rules at the end of which each person receives some benefit or suffers loss. The set of rules defines the game. Going through the set of rules once by the participants defines a play.

2. Properties of a Game

a) There are finite numbers of competitors called 'players'
b) Each player has a finite number of possible courses of action called 'strategies'
c) All the strategies and their effects are known to the players but player does not know which strategy is to be chosen.
d) A game is played when each player chooses one of his strategies. The strategies are assumed to be made simultaneously with an outcome such that no player knows his opponents strategy until he decides his own strategy.
e) The game is a combination of the strategies and in certain units which determines the gain or loss.
f) The figures shown as the outcomes of strategies in a matrix form are called 'pay-off matrix'.
g) The player playing the game always tries to choose the best course of action which results in optimal pay off called 'optimal strategy'.
h) The expected pay off when all the players of the game follow their optimal strategies is known as 'value of the game'. The main objective of a problem of a game is to find the value of the game.

i) The game is said to be 'fair' game if the value of the game is zero otherwise it s known as 'unfair'.

3. Characteristics of Game Theory

a) Competitive game: A competitive situation is called a competitive game if it has the following four properties
   - There are finite number of competitors such that $n \geq 2$. In case $n = 2$, it is called a two-person game and in case $n > 2$, it is referred as n-person game.
   - Each player has a list of finite number of possible activities.
   - A play is said to occur when each player chooses one of his activities. The choices are assumed to be made simultaneously i.e. no player knows the choice of the other until he has decided on his own.
   - Every combination of activities determines an outcome which results in a gain of payments to each player, provided each player is playing uncompromisingly to get as much as possible. Negative gain implies the loss of same amount.

b) Strategy: The strategy of a player is the predetermined rule by which player decides his course of action from his own list during the game.

   The two types of strategy are
   - Pure strategy
   - Mixed strategy

   Pure Strategy: If a player knows exactly what the other player is going to do, a deterministic situation is obtained and objective function is to maximize the gain. Therefore, the pure strategy is a decision rule always to select a particular course of action.

   Mixed Strategy: If a player is guessing as to which activity is to be selected by the other on any particular occasion, a probabilistic situation is obtained and objective function is to maximize the expected gain. Thus the mixed strategy is a selection among pure strategies with fixed probabilities.

c) Number of persons: A game is called 'n' person game if the number of persons playing is 'n'. The person means an individual or a group aiming at a particular objective.

Two-person, zero-sum game: A game with only two players (player A and player B) is called a 'two-person, zero-sum game', if the losses of one player are equivalent to the gains of the other so that the sum of their net gains is zero.

Two-person, zero-sum games are also called rectangular games as these are usually represented by a payoff matrix in a rectangular form.

d) Number of activities: The activities may be finite or infinite.

e) Payoff: The quantitative measure of satisfaction a person gets at the end of each play is called a payoff

f) Payoff matrix: Suppose the player A has 'm' activities and the player B has 'n' activities. Then a payoff matrix can be formed by adopting the following rules

- Row designations for each matrix are the activities available to player A
- Column designations for each matrix are the activities available to player B
- Cell entry Vij is the payment to player A in A's payoff matrix when A chooses the activity i and B chooses the activity j.
- With a zero-sum, two-person game, the cell entry in the player B's payoff matrix will be negative of the corresponding cell entry Vij in the player A's payoff matrix so that sum of payoff matrices for player A and player B is ultimately zero.

g) Value of the game: Value of the game is the maximum guaranteed game to player A (maximizing player) if both the players uses their best strategies. It is generally denoted by 'V' and it is unique.

4. Classification of Games

All games are classified into

- Pure strategy games
- Mixed strategy games

Strategy: It is the pre-determined rule by which each player decides his course of action from his list available to him. How one course of action is selected out of various courses available to him is known as strategy of the game.

Types of Strategy: Generally two types of strategy are employed

(i) Pure Strategy

(ii) Mixed Strategy

(i) Pure Strategy: It is the predetermined course of action to be employed by the player. The players knew it in advance. It is usually represented by a number with which the course of action is associated.

(ii) Mixed Strategy: In mixed strategy the player decides his course of action in accordance with some fixed probability distribution. Probability are associated with each course of action and the selection is done as per these probabilities.

In mixed strategy the opponent cannot be sure of the course of action to be taken on any particular occasion. Pure strategy games can be solved by saddle point method.

Decision of a Game. In Game theory, best strategy for each player is determined on the basis of some rule. Since both the players are expected to be rational in their approach this is known as the criteria of optimality. Each player lists the possible outcomes from his action and selects the best action to achieve his objectives. This criteria of optimality is expressed as Maximin for the maximising player and Minimax for the minimising player.

5. The Maximin-Minimax Principle

a) Maximin Criteria: The maximising player lists his minimum gains from each strategy and selects the strategy which gives the maximum out of these minimum gains.

b) Minimax Criteria : The minimising player lists his maximum loss from each strategy and selects the strategy which gives him the minimum loss out of these maximum losses.

For Example Consider a two person zero sum game involving the set of pure strategy for Maximising player A say Al A2 & As and for player B, B1 & B2, with the following payoff

Player B

		B1	B2	**Row MaxMin**
	A1	9	2	2
Player A	A2	8	6	6* Maximin
	A3	6	4	4
	Column MiniMax	9	6* Minimax	

Since Maximin = Minimax V = 6

Suppose that player A starts the game knowing fully well that whatever strategy he adopts B will select that particular counter strategy which will minimise the payoff to A. If A selects the strategy Al then B will select B2 so that A may get minimum gain. Similarly if A chooses A2 than B will adopt the strategy of B2. Naturally A would like to maximise his maximin gain which is just the largest of row minima. Which is called 'maximin strategy'. Similarly B will minimise his minimum loss which is called 'minimax strategy'. We observe that in the above example, the maximum if row minima and minimum of column maxima are equal. In symbols.

Maxi [Min.] = Mini [Max]

The strategies followed by both the p1ayersarecalled ‘optimum strategy’.

Value of Game. In game theory, the concept value of game is considered as very important. The value of game is the maximum guaranted gain to the maximising player if both the players use their best strategy. It refers to the average payoff per play of the game over a period of time. Consider the following the games.

$$\begin{array}{cc} & \text{Player Y} \\ \text{Player X} & \begin{pmatrix} 3 & 4 \\ -6 & -2 \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \text{Player Y} \\ \text{Player X} & \begin{pmatrix} -7 & 2 \\ -3 & -1 \end{pmatrix} \end{array}$$

(With Positive Value) (With Negative Value)

In the first game player X wins 3 points and the value of the value is three with positive sign and in the second game the player Y wins 3 points and the value of the game is -ve which indicates that Y is the Winner. The value is denoted by 'v.

The different methods for solving a mixed strategy game are

- Analytical method
- Graphical method
- Dominance rule

6. Two-Person and Zero-Sum Game

Two-person zero-sum games may be deterministic or probabilistic. The deterministic games will have saddle points and pure strategies exist in such games. In contrast, the probabilistic games will have no saddle points and mixed strategies are taken with the help of probabilities.

Definition of saddle point

A saddle point of a matrix is the position of such an element in the payoff matrix, which is minimum in its row and the maximum in its column.

Procedure to find the saddle point

- Select the minimum element of each row of the payoff matrix and mark them with circles.
- Select the maximum element of each column of the payoff matrix and mark them with squares.
- If their appears an element in the payoff matrix with a circle and a square together then that position is called saddle point and the element is the value of the game.

Solution of games with saddle point

To obtain a solution of a game with a saddle point, it is feasible to find out

- Best strategy for player A
- Best strategy for player B
- The value of the game

The best strategies for player A and B will be those which correspond to the row and column respectively through the saddle point.

Example 1: Solve the payoff matrix

Player B

		I	II	III	IV	V
	I	-2	0	0	5	3
	II	3	2	1	2	2
Player A	III	-4	-3	0	-2	6
	IV	5	3	-4	2	-6

Solution:

Player B

		I	II	III	IV	V	Row MaxMin	
	I	-2	0	0	5	3	-2	
Player A	II	3	2	$\boxed{1}$	2	2	1	Maxmin Value
	III	-4	-3	0	-2	6	-4	

IV	5	3	-4	2	-6	-6
Column MiniMax	5	3	1	5	6	

Minimax value

Strategy of player A – II

Strategy of player B - III

Value of the game = 1

Example -2: Solve the payoff matrix

	B1	**B2**	**B3**	**B4**
A1	1	7	3	4
A2	5	6	4	5
A3	7	2	0	3

Solution

	B1	**B2**	**B3**	**B4**	**Row MaxMin**	
A1	1	7	3	4	**1**	
A2	5	6	$\boxed{4}$	5	**4**	**Maximin Value**
A3	7	2	0	3	**0**	
Column MiniMax	**7**	**7**	**4**	**5**		

Minimax Value

Strategy of player A= A2

Strategy of player B = B3

Value of the game = 4

Points to remember

(i) Saddle point mayor may not exist in a given game.

(ii) There may be more than one saddle point then there will be more than one solution. (Such situation is rare in the rea1life).

(iii) The value of game may be +ve or -ve.

(iv) The value of game may be zero which means 'fair game'.

II GAMES WITH MIXED STRATEGIES

All game problems, where saddle point does not exist are taken as mixed strategy problems. Where row minima is not equal to column maxima, then different methods are used to solve the different types of problems. Both players will use different strategies with certain probabilities to optimise. For the solution of games with mixed strategies, any of the following methods can be applied.

1. ODDS METHOD (2x2 game without saddle point)
2. Dominance Method.
3. Sub Games Method. – For (mx2) or (2xn) Matrices
4. Equal Gains Method.
5. Linear Programming Method-Graphic solution

These methods are explained one by one with examples, in detail.

1. ODDS Method - For 2 x 2 Game

Use of odds method is possible only in case of games with 2 x 2 matrix. Here it should be ensured that the sum of column odds and row odds is equal.

METHOD OF FINDING OUT ODDS

Step 1: Find out the difference in the value of in cell (1, 1) and the value in the cell (1,2) of the first row and place it in front of second row.

Step 2: Find out the difference in the value of cell (2, 1) and (2, 2) of the second row and place it in front of first row.

Step 3: Find out the differences in the value of cell (1, 1) and (2, 1) of the first column and place it below the second column.

Step 4: Similarly find the difference between the value of the cell (1, 2) and the value in cell (2, 2) of the second column and place it below the first column.

The above odds or differences are taken as positive (ignoring the negative sign)

Strategy →		Y		
↓		Y_1	Y_2	ODDS
X	X_1	a_1	a_2	→ $(b_1 - b_2)$
	X_2	b_1	b_2	→ $(a_1 - a_2)$
ODDS		↓ $(a_2 - b_2)$	↓ $(a_1 - b_1)$	

The value of game is determined with the help of following equation.

$$\text{Value of the game } (\upsilon) = \frac{a_1(b_1 - b_2) + b_1(a_1 - a_2)}{(b_1 - b_2) + (a_1 - a_2)}$$

$$\text{Probabilities for } X_1 = \frac{b_1 - b_2}{(b_1 - b_2) + (a_1 - a_2)}, \quad X_2 = \frac{a_1 - a_2}{(b_1 - b_2) + (a_1 - a_2)}$$

$$\text{Probabilities for } Y_1 = \frac{a_2 - b_2}{(a_2 - b_2) + (a_1 - b_1)}, \quad Y_2 = \frac{a_1 - b_1}{(a_2 - b_2) + (a_1 - b_1)}$$

Strategy		Player B	
		B_1	B_2
Player A	A_1	1	5
	A_2	4	2

Solution

		Player B	
		B_1	B_2
Player A	A_1	1	$\boxed{5}$
	A_2	4	2

Since the game does not have saddle point, the players will use mixed strategy. We apply odds methods to solve the game.

	B_1	B_2	odds
A_1	1	5	$4 - 2 = 2$
A_2	4	2	$1 - 5 = 4$
Odds	$5 - 2 = 3$	$1 - 4 = 3$	

$$\text{Value of the game } (\upsilon) = \frac{(1\times 2)+(4\times 4)}{2+4} = 3, \qquad \upsilon = 3$$

Probabilities of Selecting Strategies

		I	II
	A	1/3	2/3
Players	B	1/2	1/2

2. **Dominance Method.** Dominance method is also applicable to pure strategy and mixed strategy problem. In pure strategy the solution is obtained by itself while in mixed strategy it can be used for simplifying the problem.

Principle of Dominance. The Principle of Dominance states that if the strategy of a player dominates over the other strategy in all condition then the later strategy is ignored because it will not effect the solution in any way. For the gainer point of view if a strategy gives more gain than another strategy, then first strategy dominates over the other and the second strategy can be ignored altogether. Similarly from loser point of view, if a strategy involves lesser loss than other in all condition then second can be ignored. So determination of superior or inferior strategy is based upon the objective of the player. Since each player is to select his best strategy, the inferior strategies can be eliminated. In other words, ineffective rows & column can be deleted from the game matrix and only effective rows & columns of the matrix are retained in the reduced matrix.

For deleting the ineffective rows & columns the following general rules are to be followed.

1) If all the elements of a row (say i^{th} row) of a pay off matrix are less than or equal to (□) the corresponding each element of the other row (say j^{th} row) then the player A will never choose the i^{th} strategy OR i^{th} row is dominated by j^{th} row. Then delete i^{th} row.

   $$\text{Eg. } E_{ij} - [R_{ith}] \leq E_{ij}\begin{pmatrix} R_{ith} \\ Row \end{pmatrix} \text{ Delete } R_{ith} \text{ rows.}$$

2) If all the elements of a column (say j^{th} column are greater than or equal to the corresponding elements of any other column (say j^{th} column) then i^{th} column is dominated by j^{th} column. Then delete i^{th} column.

   $$E_{ij}(c_i) \geq E_{ij}(cj)$$
   $$\text{delete} = c_i \text{ th}$$

3) A pure strategy of a player may also be dominated if it is inferior to some convex combination of two or more pure strategies. As a particular case, if all the elements of a column are greater than or equal to the average of two or more other columns then this column is dominated by the group of columns. Similarly if all the elements of row are less than or equal to the average of two or more rows then this row is dominated by other group of row.

4) By eliminating some of the dominated rows a columns and if the game is reduced to 2 x 2 form it can be easily solved by odds method.

Example: Solve the game.

B

A			
	5	20	-10
	10	6	2
	20	15	18

A \ B	I	II	III
I	5	$\boxed{20}$	-10
II	10	6	2
III	$\boxed{20}$	15	$\boxed{18}$

Since there is no saddle point, so we apply dominance method. Here Row II dominates Row II so we will delete Row II.

A \ B	I	II	III
I	5	20	-10
III	20	15	18

Since column III dominates column I, we delete column I we get:

A \ B	II	III	odds
I	20	-10	3
III	15	18	30
Odds	28	5	33

$$\text{Value of the game } = \frac{20(3)+15(30)}{3+30} = \frac{510}{33} = \frac{170}{11}$$

Probability of Selecting Strategies

Player A
I = 3 / 33 = 1/ 11
II = 0
III = 30/33 = 10/11

Player B
I = 0
II = 28 / 33
III = 5/33

Using the dominance property obtain the optimal strategies for both the players and determine the value of the game. The pay off matrix for player A is given.

Example 2:

		Player B				
		I	II	III	IV	V
Player A	I	2	4	3	8	4
	II	5	6	3	7	8
	III	6	7	9	8	7
	IV	4	2	8	4	3

Solution.

Since the questions desires to show the dominance property, that is why we are using it, even if the question may have saddle point.

Since all elements in Row IV are less than respective each element in Row III. Row III dominates Row IV. Hence we delete Row IV & we get

		Player B				
		I	II	III	IV	V
Player A	I	2	4	3	8	4
	II	5	6	3	7	8
	III	6	7	9	8	7

Now all the elements of column I are less than or equal to respective each elements of column IV, so we can delete column IV now we get

		Player B			
		I	II	III	V
Player A	I	2	4	3	4
	II	5	6	3	8
	III	6	7	9	7

Repeating the above rules now each element of column I is less than the respective elements of column V, we can delete the column V and we get.

	B I	B II	B III
I	2	4	3
II	5	6	3
III	6	7	9

Now as each element of Row I and Row II are less than the respective elements of III, we can delete both Rows I & II

		Player B I	Player B II	Player B III
Player A	III	6	7	9

Value of Game (v) = 6

Strategy for player A $\rightarrow$ III

Strategy for player B $\rightarrow$ I

Graphical method

The graphical method is used to solve the games whose payoff matrix has

- Two rows and n columns (2 x n)
- m rows and two columns (m x 2)

Algorithm for solving 2 x n matrix games

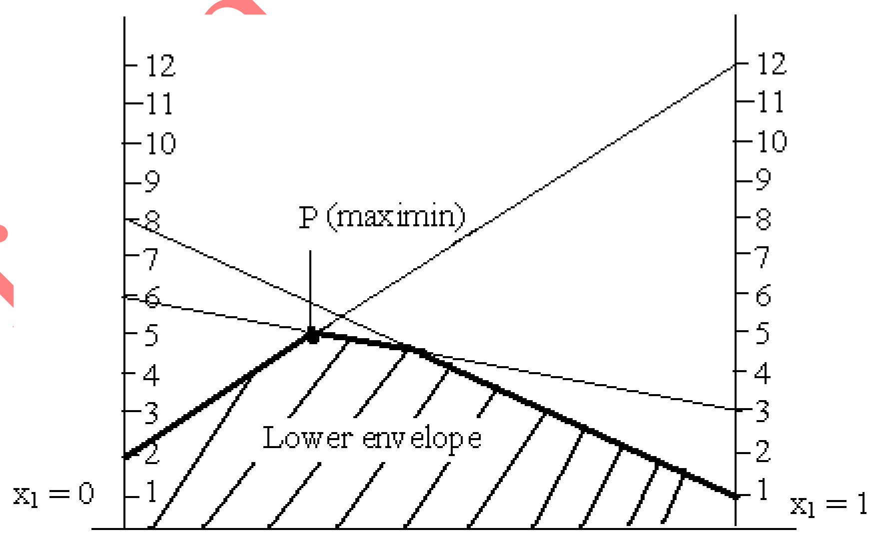
- Draw two vertical axes 1 unit apart. The two lines are x1 = 0, x1 = 1
- Take the points of the first row in the payoff matrix on the vertical line x1 = 1 and the points of the second row in the payoff matrix on the vertical line x1 = 0.
- The point a1j on axis x1 = 1 is then joined to the point a2j on the axis x1 = 0 to give a straight line. Draw 'n' straight lines for j=1, 2… n and determine the highest point of the lower envelope obtained. This will be the maximin point.
- The two or more lines passing through the maximin point determines the required 2 x 2 payoff matrix. This in turn gives the optimum solution by making use of analytical method.

Example 1

Solve by graphical method

$$\begin{array}{c|ccc} & B1 & B2 & B3 \\ \hline A1 & 1 & 3 & 12 \\ A2 & 8 & 6 & 2 \end{array}$$

Solution:



$$\begin{array}{cccc} & B2 & B3 & \\ A1 & \left[\begin{array}{c} 3 \\ 6 \end{array}\right. & \left.\begin{array}{c} 12 \\ 2 \end{array}\right] & \begin{array}{c} 4 \\ 9 \end{array} \\ & 10 & 3 & \end{array}$$

$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{6 - 72}{5 - 18}$$

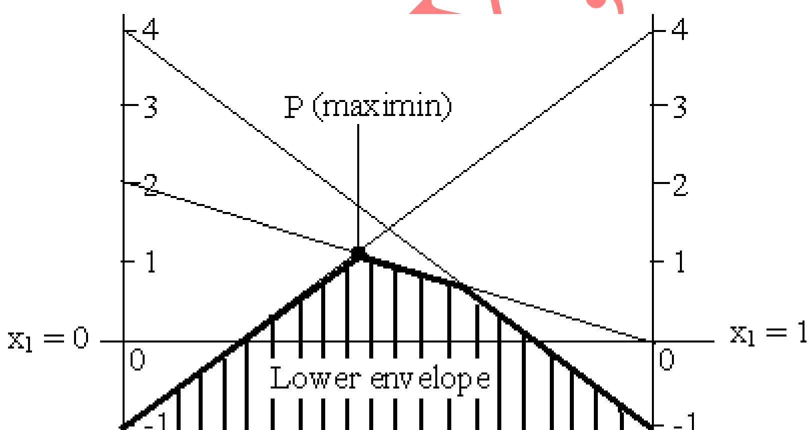
V = 66/13
S_A = (4/13, 9 /13)
S_B = (0, 10/13, 3 /13)

Example 2

Solve by graphical method

$$\begin{array}{cccc} & B1 & B2 & B3 \\ \begin{array}{c} A1 \\ A2 \end{array} & \left[\begin{array}{c} 4 \\ -1 \end{array}\right. & \begin{array}{c} -1 \\ 4 \end{array} & \left.\begin{array}{c} 0 \\ 2 \end{array}\right] \end{array}$$

Solution:



$$\begin{array}{cccc} & B1 & B3 & \\ \begin{array}{c} A1 \\ A2 \end{array} & \left[\begin{array}{c} 4 \\ -1 \end{array}\right. & \left.\begin{array}{c} 0 \\ 2 \end{array}\right] & \begin{array}{c} 3 \\ 4 \end{array} \\ & 2 & 5 & \end{array}$$

$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{8 - 0}{6 + 1}$$

$V = 8/7$
$S_A = (3/7, 4/7)$
$S_B = (2/7, 0, 5/7)$

Algorithm for solving m x 2 matrix games

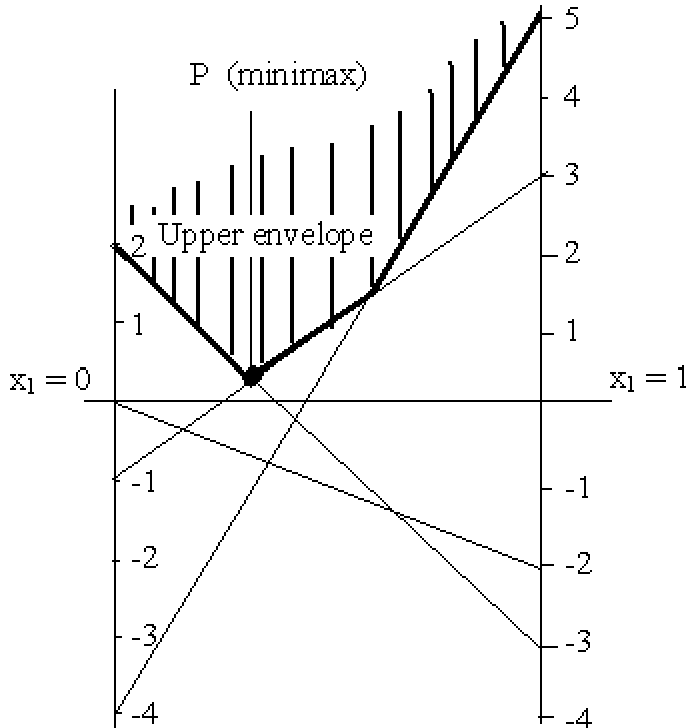
- Draw two vertical axes 1 unit apart. The two lines are x1 =0, x1 = 1
- Take the points of the first row in the payoff matrix on the vertical line x1 = 1 and the points of the second row in the payoff matrix on the vertical line x1 = 0.
- The point a1j on axis x1 = 1 is then joined to the point a2j on the axis x1 = 0 to give a straight line. Draw 'n' straight lines for j=1, 2… n and determine the lowest point of the upper envelope obtained. This will be the minimax point.
- The two or more lines passing through the minimax point determines the required 2 x 2 payoff matrix. This in turn gives the optimum solution by making use of analytical method.

Example 1

Solve by graphical method

$$\begin{array}{cc} & \begin{array}{cc} B1 & B2 \end{array} \\ \begin{array}{c} A1 \\ A2 \\ A3 \\ A4 \end{array} & \begin{bmatrix} -2 & 0 \\ 3 & -1 \\ -3 & 2 \\ 5 & -4 \end{bmatrix} \end{array}$$

Solution



$$\begin{array}{c|cc|c} & B1 & B2 & \\ \hline A2 & 3 & -1 & 5 \\ A3 & -3 & 2 & 4 \\ \hline & 3 & 6 & \end{array}$$

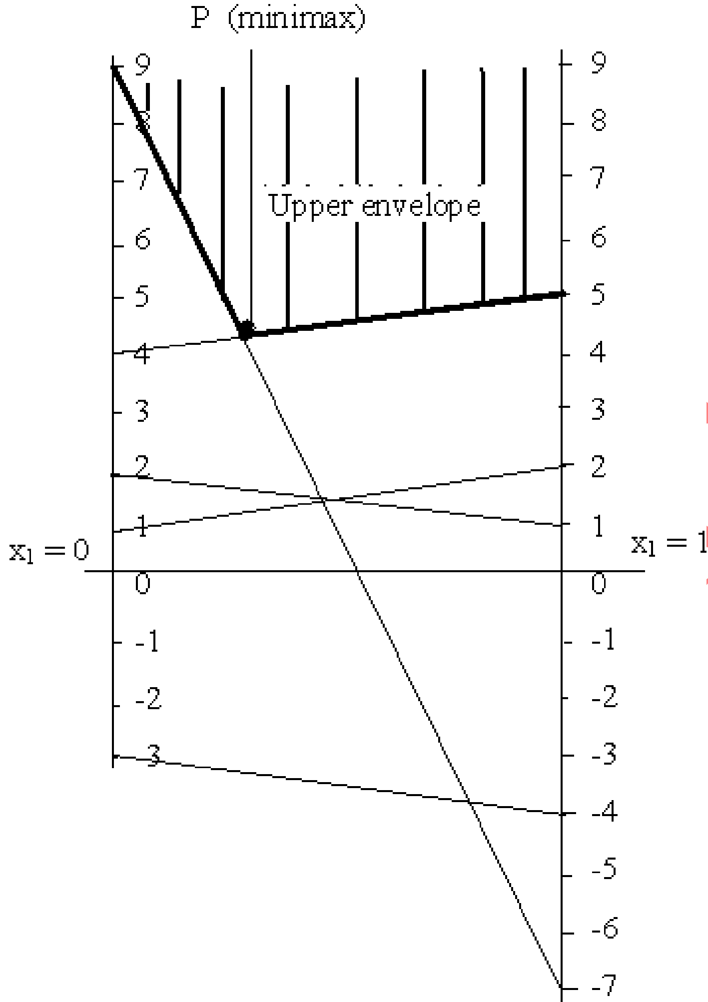
$$V = \frac{a_{11}\, a_{22} - a_{21}\, a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{6-3}{5+4}$$

V = 3/9 = 1/3
S_A = (0, 5 /9,
4/9, 0) S_B =
(3/9, 6 /9)

Example 2: Solve by graphical method

$$\begin{array}{c} \\ A1 \\ A2 \\ A3 \\ A4 \\ A5 \end{array} \begin{array}{c} \begin{array}{cc} B1 & B2 \end{array} \\ \begin{bmatrix} 1 & 2 \\ 5 & 4 \\ -7 & 9 \\ -4 & -3 \\ 2 & 1 \end{bmatrix} \end{array}$$

Solution:



$$
\begin{array}{c|cc|c}
 & B1 & B2 & \\
\hline
A2 & 5 & 4 & 16 \\
A3 & -7 & 9 & 1 \\
\hline
 & 5 & 12 &
\end{array}
$$

$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{45 + 28}{14 + 3}$$

V = 73/17
S_A = (0, 16/17, 1/17, 0, 0)
S_B = (5/17, 12 /17)