

Module - IISimplex Method - I

To solve a linear programming problem graphical method is used when only two decision variable are present. But ~~when~~ most real life problems when formulated as LP model will have more than two decision variables. A "Simplex Method" is used for solving LPP with a larger number of variables.

The Essence of Simplex method

Simplex method is an algebraic procedure. Its underlying concepts are geometric.

The geometric concepts are related to the algebra of the Simplex method. In graphical method of solving an LPP, we used to identify a common region known as feasible region satisfying all the constraints. The optimal solution used to occur at some vertex of the feasible region.

If the optimal solution was not unique, the optimal points were on an edge. Essentially the problem is that of finding the particular vertex for the convex region which corresponds to the optimal solution.

The Simplex method provides an algorithm which consists in moving from one vertex of the region of feasible solution to another vertex in such a way that the value of the objective function at the succeeding vertex is improved than at the previous vertex.

This procedure of iteration from one vertex to another is repeated till the optimal solution is obtained. Thus, the geometric concepts are related to the algebra on which simplex method works.

BASIC TERMS / DEFINITIONS

Slack Variable: A variable added to the left hand side of a constraint (less than or equal to) to convert the constraint into an equation is called a slack variable.

Eg: If the constraint given is $3x_1 + 5x_2 \leq 10$ then $3x_1 + 5x_2 + s_1 = 10$ is the equality (equation) where s_1 is a slack variable.

Surplus Variable: A variable subtracted from the left hand side of the constraint (greater than or equal to) and to convert it into an equality is called a surplus variable.

Eg: If the constraint given is $5x_1 + 8x_2 \geq 12$ then $5x_1 + 8x_2 - s_1 = 12$ is the equation. where s_1 is the surplus variable.

Basic Solution: The initial solution obtained after setting the basic variables at zeros is basic solution. It is the unique solution resulting from setting $(n-m)$ variables equal to zero.

Where,

m = number of simultaneous linear equations

n = number of variables.

Basic feasible solution: A basic solution which satisfies $x_i \geq 0$, $i = 1, 2, \dots, n$ is called a basic feasible solution.

Optimal Solution: Any basic feasible solution which optimizes (minimizes or maximizes) the objective function of a general LP problem is known as an optimal solution.

Standard form of an LP Problem (Characteristics of LPF)
The standard form of the LP problem should have the following characteristics.

- (i) All the constraints should be expressed as equations by adding slack or surplus variables.
- (ii) The right hand side of each constraint should be made non negative. if it is not, this should be done by multiplying both sides of the resulting constraint by -1.
- (iii) The objective function should be of the maximization type (if it is not, should be converted by multiplying with -1).

(eg) obtain all the basic solutions to the following system of linear equations. Is the non-degenerate solution feasible?

$$2x_1 + 6x_2 + 2x_3 + x_4 = 3$$

$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$$

which of them are basic feasible solutions.

sol:

Number of unknown (variables) = 4

Number of equations = 2

There will be $4C_2 (= 6)$ different possible basic solutions.

S.No	Basic Variable	Non-Basic Variable	Value of basic Variables	Is the solution feasible?
1	x_1, x_2	$x_3 = x_4 = 0$	$2x_1 + 6x_2 = 3$ $6x_1 + 4x_2 = 2$ $x_1 = 0, x_2 = \frac{1}{2}$	Yes
2	x_1, x_3	$x_2 = x_4 = 0$	$2x_1 + 2x_3 = 3$ $6x_1 + 4x_3 = 2$ $x_1 = -2, x_3 = \frac{7}{2}$	No
3	x_1, x_4	$x_2 = x_3 = 0$	$2x_1 + x_4 = 3$ $6x_1 + 6x_4 = 2$ $x_1 = \frac{8}{4}, x_4 = -\frac{7}{3}$	No
4	x_2, x_3	$x_1 = x_4 = 0$	$6x_2 + 2x_3 = 3$ $4x_2 + 4x_3 = 2$ $x_2 = \frac{1}{2}, x_3 = 0$	Yes
5	x_2, x_4	$x_1 = x_3 = 0$	$6x_2 + x_4 = 3$ $4x_2 + 6x_4 = 4$ $x_2 = \frac{1}{2}, x_4 = 0$	Yes
6	x_3, x_4	$x_1 = x_2 = 0$	$2x_3 + x_4 = 3$ $4x_3 + 6x_4 = 2$ $x_3 = 2, x_4 = -1$	No

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Eg

Find all the basic solutions of the following system of equations identifying in each case the basic & nonbasic variables

$$2x_1 + x_2 + 4x_3 = 1$$

$$3x_1 + x_2 + 5x_3 = 14$$

Sol:

No. of equations = 2

No. of variables = 3

∴ There are $3C_2$ possible ways of getting different basic solutions $3C_2 = \frac{3 \times 2}{2} = 3$ ways.

When, $x_3 = 0$,

$$2x_1 + x_2 = 11$$

on solving

$$x_1 = 0$$

$$x_2 + 4x_3 = 11$$

on solving

$$x_2 = 0$$

$$2x_1 + 4x_3 = 11$$

on solving

$$3x_1 + x_2 = 14$$

$$x_1 = 3, x_2 = 5$$

$$x_2 + 5x_3 = 14$$

$$x_2 = -1, x_3 = 3$$

$$3x_1 + 5x_3 = 14$$

$$x_1 = \frac{1}{2}, x_3 = \frac{5}{2}$$

S.No	Basic Variables	Non-basic Variables	Value of basic Variables	Is it sol? feasible
1	x_1, x_2	x_3	$x_1 = 3, x_2 = 5$	Yes
2	x_2, x_3	x_1	$x_2 = 2, x_3 = 1$	No
3	x_1, x_3	x_2	$x_1 = \frac{1}{2}, x_3 = \frac{5}{2}$	Yes

Infeasible solution: If the basic variable values are negative the solution is stated as infeasible.

The setting up & Algebra of Simplex Method

Step 1: Check whether the given objective function is to be maximized or minimized.

If the objective function is to be maximized, take it in the given form itself. If it is to be minimized then convert into maximization form (by multiplying the coefficients of objective func with -ve sign)

Step 2: Express the problem in the standard form by introducing appropriate slack/surplus variables.

Convert each constraint to an equation.

Step 3: Set up the first starting solution.

Step 4: Check solution for optimality. If optimal, stop otherwise continue.

Step 5: Select a variable to enter to improve the solution.

Step 6: Select a variable to leave the basis.

Step 7: Perform row operations to complete the solution.

Step 8: Return to Step 3 & continue the procedure until optimality is obtained.

Outline of Simplex method:

S₁: Determine the starting basic feasible solution.

S₂: Select an entering variable using row operations.
If there is no entering variable, the solution is optimal & stop process. Else go to Step 3.

S₃: Select a leaving variable using the row operations based on ratio.

S₄: Determine the new basic solution using the row operation. Go to Step 2.

Steps in performing row operations

- i) Identify the pivot element (PE) of the pivotal row (PR) & pivotal column (PC).
- ii) Divide the PR, element by element, by the PE. Enter this new row in the next tableau in the same row position.
- iii) Reduce all other elements in the PC in the next tableau to zeros by multiplying the new row formed in Step 2 by the negative of the row's elements in the PC of the present tableau and add it this transitioned row to the row being modified.
- iv) New pivot equation = $\frac{\text{old equation element}}{\text{pivot element}}$ and the other equations above or below to the pivot element equation = old element - [its existing coefficient of the column \times corresponding new pivot equation].

$$L = L - (1 \times 0 + 2 \times 0)$$

$$B = B - (2 \times 0 + 1 \times 0)$$

(Q) Solve the following LP problem by the Simplex method

$$\text{Maximize } Z = 2x_1 + 3x_2$$

Subject to the constraint

$$2x_1 + x_2 \leq 12$$

$$x_1 + 3x_2 \leq 15$$

$$x_1 \geq 0, x_2 \geq 0.$$

Sol: The objective function is in the standard form (Maximization) & hence there is no need for modification.

$$Z_{\max} = 2x_1 + 3x_2$$

Converting the inequalities as equations we get

$$2x_1 + x_2 + s_1 = 12$$

$$x_1 + 3x_2 + s_2 = 15$$

In terms of $x_1, x_2, s_1, s_2 \geq 0$ all these variables the objective func is write in the form of table

$$Z_{\max} = 2x_1 + 3x_2 + 0s_1 + 0s_2$$

		2	3	0	0	Ratio (B/Element of KC)
B	x_1	x_2	s_1	s_2		$12/x_1 = 12$
0	s_1	12	2	1	0	
0	s_2	15	1	3	0	$15/3 = 5$
	Z	0	-2	-3	0	Pivot Element

Index row → Key row
Key column → Pivot Element

Calculate value of Z:

$$(0*2 + 0*1) - 2 = -2$$

$$(0*1 + 0*3) - 3 = -3$$

⑤

$$(0+1+0+0) - 0 = 0$$

$$(0+1+0+1) - 0 = 1$$

We got the value of Z . Next find the maximum negative value in Z to identify key column

(Here x_2 is key column because x_2 has highest negative value)

key row: Divide key column by the basic column (B)

(i.e. $12/1 = 12$ & $15/3 = 5$) The result which has least positive element is going to be key row

key Element: Key element is one where key row & key column intersect

[i.e. in this eg 3 is key element]

Next step is to make key element 1 to make key one in this example ^{multiply} divided S_2 row by $\frac{1}{3}$

$$\frac{1}{3} S_2 \quad 15 \times \frac{1}{3} \quad 1 \times \frac{1}{3} \quad 8 + \frac{1}{3} \quad 0 + \frac{1}{3} \quad 1 \times \frac{1}{3}$$

$$\frac{1}{3} S_2 \quad 5 \quad \frac{1}{3} \quad 1 \quad 0 \quad \frac{1}{3}$$

Next table [S_2 leaves the table & x_2 entries]
(key column)

$$R_1 \quad 0 \quad S_1 \quad 12 \quad 2 \quad \boxed{1} \quad 1 \quad 0$$

$$R_2 \quad 3 \quad x_2 \quad 5 \quad \frac{1}{3} \quad \boxed{1} \quad 0 \quad \frac{1}{3}$$

Other Element in key column should be made zero

$$\therefore R'_1 = R_1 - R_2$$

R'_1 = old value of S_1

R'_2 = new value of $S_2(x_2)$.

$$R_1' = R_1 - R_2$$

$$R_1' = (12-5) \quad (2-\frac{1}{3}) \quad (1-1) \quad (r-0) \quad (0-\frac{1}{3})$$

$$S_1 = 7 \quad \frac{6-1}{3} = \frac{5}{3} \quad 0 \quad 1 \quad -\frac{1}{3}$$

2 3 0 0

0	S_1	7	$\frac{5}{3}$	0	1	$-\frac{1}{3}$	$\frac{7}{5} = \frac{7 \times 3}{5} = \frac{21}{5} = 4.2$
3	x_2	5	$\frac{1}{3}$	1	0	$\frac{1}{3}$	$\frac{5}{1} = \frac{5 \times 3}{1} = \frac{15}{1} = 15$
2		15	-1	0	0	1	

Calculate the value of Z

$$(0 \times 7 + 3 \times 5) = 0 + 15 = 15$$

$$(0 \times \frac{5}{3} + 3 \times \frac{1}{3}) - 2 = (0 + \frac{3}{3}) - 2 = 1 - 2 = -1$$

$$(0 \times 0 + 1 \times 3) - 3 = (0 + 3) - 3 = 3 - 3 = 0$$

$$(1 \times 0 + 0 \times 3) - 0 = 0$$

$$(0 \times -\frac{1}{3} + 3 \times \frac{1}{3}) - 0 = (0 + \frac{3}{3}) - 0 = 1 - 0 = 1$$

To make key element 1 multiply S_1 by $\frac{3}{5}$

$$\frac{3}{5} \times S_1 \quad 7 \times \frac{3}{5} \quad \frac{5}{8} \times \frac{3}{5} \quad 0 \times \frac{3}{5} \quad 1 \times \frac{3}{5} = -\frac{1}{2} \times \frac{3}{5}$$

$$\boxed{\frac{3}{5} S_1 \quad \frac{21}{5} \quad 1 \quad 0 \quad \frac{3}{5} \quad -\frac{1}{2} \times \frac{3}{5}}$$

Now S_1 leaves x_1 . Enter the table

$$\boxed{2 x_1 \quad \frac{21}{5} \quad 1 \quad 0 \quad \frac{3}{5} \quad -\frac{1}{2} \times \frac{3}{5}}$$

In the key element column, other element in that particular column should be made zero. Here in this example x_2 i.e. $\frac{1}{3}$ should be made zero to make $\frac{1}{3}$ zero we should apply R_2' i.e new second row
 $x_2 = R_2$ (old x_2 row) - $\frac{1}{3} R_1$ (new x_1 row)

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$$R_2' = R_2 - \frac{1}{3} R_1$$

$$3 \ x_2 \ 5 \ \frac{1}{3} \ 1 \ 0 \ \frac{1}{3} \ || R_2 \text{ (old } x_2 \text{ row)}$$

$$\begin{aligned} R_2' &= 5 - \frac{1}{3} \times \frac{21}{5} & \frac{1}{3} \times -\frac{1}{3} & 1 - \frac{1}{3} & 0 + \frac{1}{3} & \frac{1}{3} + \frac{1}{3} \\ R_2' &= 5 - \frac{1}{3} \times \frac{21}{5} & \frac{1}{3} - \frac{1}{3} & 1 - \frac{1}{3} & 0 - \frac{1}{3} & \frac{1}{3} - \frac{1}{3} \\ R_2' &= 5 - \frac{7}{5} & 0 & 1 & 0 - \frac{1}{5} & \frac{6}{15} \\ &= \frac{18}{5} & 0 & 1 & -\frac{1}{5} & \frac{6}{15} \end{aligned}$$

Now table be

$$\begin{array}{ccccccc} 2 & x_1 & \frac{21}{5} & 1 & 0 & \frac{3}{5} & -\frac{1}{5} \\ 3 & x_2 & \frac{18}{5} & 0 & 1 & -\frac{1}{5} & \frac{2}{5} \\ \hline 2 & \frac{96}{5} & 0 & 0 & \frac{3}{5} & \frac{4}{5} \end{array}$$

Since all the index row elements are +ve stop
the process

$$\text{we find } x_1 = \frac{21}{5} \text{ & } x_2 = \frac{18}{5}$$

Hence the optimal solution (that is Z_{\max})
is

$$\begin{aligned} Z_{\max} &= 2 \times \frac{21}{5} + 3 \times \frac{18}{5} \\ &= \frac{42}{5} + \frac{54}{5} \\ &= \underline{\underline{\frac{96}{5}}} \end{aligned}$$

(Q) Solve the following LP problem by the simplex method.

$$\text{Maximize } Z = 6x_1 + 8x_2$$

Subject to the constraints

$$5x_1 + 10x_2 \leq 60$$

$$4x_1 + 4x_2 \leq 40$$

$$x_1, x_2 \geq 0$$

Sol:

$$Z_{\max} = 6x_1 + 8x_2 + 0s_1 + 0s_2$$

$$5x_1 + 10x_2 + s_1 = 60$$

$$4x_1 + 4x_2 + s_2 = 40$$

$$x_1, x_2, s_1, s_2 \geq 0$$

Initial table

			6	8	0	0
	Basic	RHS	x_1	x_2	s_1	s_2
R_1	0	s_1	60	5	10	0
R_2	0	s_2	40	4	4	1
			2	-6	-8	0
	Z		0	-6	-8	0

Divide s_1 row by 10 so that key element becomes 1

$$\frac{1}{10} s_1 \quad 6 \quad \frac{1}{2} \quad 1 \quad \frac{1}{10} \quad 0$$

s_1 leave & x_2 enter

$$R_1' \quad 8 \quad x_2 \quad 6 \quad \frac{1}{2} \quad 1 \quad \frac{1}{10} \quad 0$$

$$R_2' = R_2 - 4R_1'$$

$$R_2' = 40 - 4 \cdot 6 \quad 4 - 4 \cdot \frac{1}{2} \quad 4 - 4 \cdot 1 \quad 0 - 4 \cdot \frac{1}{10} \quad 1 - 4 \cdot 0$$

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	Basic variable	RHS	x_1	x_2	s_1	s_2	
R_1	x_2	6	$\frac{1}{2}$	1	$\frac{1}{10}$	0	$6 \times \frac{1}{2} = \frac{6}{2} + \frac{2}{2} = 12$
0	x_1	16	2	0	$-\frac{2}{5}$	1	$16 \times \frac{1}{2} = 8$
	x_1	48	-2	0	$\frac{4}{5}$	0	

To make key element 1 multiply by $\frac{1}{2}$ to x_1 row.

	x_1	$16 \times \frac{1}{2}$	$2 \times \frac{1}{2}$	$0 \times \frac{1}{2}$	$-\frac{2}{5} \times \frac{1}{2}$	$1 \times \frac{1}{2}$	
R_2	x_1	8	1	0	$-\frac{1}{5}$	$\frac{1}{2}$	

$$\text{Appy } R'_1 = R_1 - \frac{1}{2} R_2$$

$$\begin{aligned}
 R'_1 &= 6 - \frac{1}{2} \times 8 & \frac{1}{2} - \frac{1}{2} \times 1 & 1 - \frac{1}{2} \times 0 & \frac{1}{10} - \frac{1}{2} \times \frac{1}{5} & 0 - \frac{1}{2} \times \frac{1}{2} \\
 &= 6 - 4 & \frac{1}{2} - \frac{1}{2} & 1 - 0 & \frac{1}{10} + \frac{1}{10} & 0 - \frac{1}{4} \\
 &= 2 & 0 & 1 & \frac{2}{10} = \frac{1}{5} & -\frac{1}{4}
 \end{aligned}$$

CB	BV	RHS	x_1	x_2	s_1	s_2	
8	x_2	2	0	1	$\frac{1}{5}$	$-\frac{1}{4}$	
6	x_1	8	1	0	$-\frac{1}{5}$	$\frac{1}{2}$	
	x_1	64	0	0	$\frac{2}{5}$	1	

All values are positive so optimal solution reached

$$\therefore x_1 = 8, x_2 = 2 \text{ & } Z = 64$$

By calculation

$$Z = 6x_1 + 8x_2$$

$$= 6 \times 8 + 8 \times 2$$

$$= 48 + 16$$

$$= \underline{\underline{64}}$$

Eg Solve the following LP problem by the simplex method

$$\text{Max } Z = 2x_1 + 2x_2 + 4x_3$$

subject to the constraint

$$2x_1 + 3x_2 + x_3 \leq 240$$

$$x_1 + x_2 + 3x_3 \leq 300$$

$$x_1 + 3x_2 + x_3 \leq 300$$

$$x_1, x_2, x_3 \geq 0$$

Sol:

$$Z_{\max} = 2x_1 + 2x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$$

$$2x_1 + 3x_2 + x_3 + s_1 = 240$$

$$x_1 + x_2 + 3x_3 + s_2 = 300$$

$$x_1 + 3x_2 + x_3 + s_3 = 300$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

			2	2	4	0	0	0
			x_1	x_2	x_3	s_1	s_2	s_3
R ₁	0	s_1	240	2	3	1	1	0
R ₂	0	s_2	300	1	1	3	0	1
R ₃	0	s_3	300	1	3	1	0	0
	Z		0	-2	-2	-4	0	0

$240/1 = 240$

$300/3 = 100$

$300/1 = 300$

R ₁	s_1	240	2	3	1	1	0	0
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R ₂ '	$s_2/3$	100	$\frac{1}{3}$	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0
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R ₃ '	s_3	300	1	3	1	0	0	1
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$$R_1' = R_1 - R_2'$$

0	s_1	$240 - 100$	$2 - \frac{1}{3}$	$3 - \frac{1}{3}$	1-1	1-0	$0 - \frac{1}{3}$	0-0
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s_1	140	$\frac{6-1}{3} = \frac{5}{3}$	$\frac{9-1}{3} = \frac{8}{3}$	0	1	$-\frac{1}{3}$	0
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$$R'_3 = R_3 - R'_2$$

$$0 \quad S_3 \quad 300 - 100 \quad 1 - \frac{1}{3} \quad 3 - \frac{1}{3} \quad 1 - 1 \quad 0 - 0 \quad 0 - \frac{1}{3} \quad 1 - 0$$

$$0 \quad S_3 \quad 200 \quad \frac{3-1}{3} = \frac{2}{3} \quad \frac{9-1}{3} = \frac{8}{3} \quad 0 \quad 0 \quad -\frac{1}{3} \quad 1$$

Table

		2	2	4	0	0	0	
		x_1	x_2	x_3	S_1	S_2	S_3	
0	S_1	140	$\frac{5}{3}$	$\frac{8}{3}$	0	1	$-\frac{1}{3}$	0
4	x_3	100	$\frac{1}{3}$	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0
0	S_3	200	$\frac{2}{3}$	$\frac{8}{3}$	0	0	$-\frac{1}{3}$	1
	Z	400	$-\frac{2}{3}$	$-\frac{2}{3}$	0	0	$\frac{4}{3}$	0

Note: Since $-\frac{2}{3}$ & $-\frac{2}{3}$ exists at two places in the index row, choose any one & find the key row then find the value of Z. Again choose another $-\frac{2}{3}$ & find Z, then select the column which resulted in maximum value of Z as the key column.

i.e. for one column value of Z = $400 - \frac{140(-\frac{2}{3})}{\frac{5}{3}} = 456$

For another column value of Z = $400 - \frac{140(-\frac{2}{3})}{\frac{8}{3}} = 435$

\therefore The first column which resulted in maximum value of Z is selected as key column.

To make key element 1 multiply key row by $\frac{3}{5}$.

$$2 \quad x_1 \quad 140 + \frac{3}{8} \quad 15 + \frac{8}{5} \quad 8 + \frac{8}{5} \quad 0 + \frac{3}{5} \quad 1 + \frac{3}{5} \quad -\frac{1}{3} + \frac{8}{5}$$

$$\begin{matrix} R_1' \\ \text{new } S_1 \end{matrix} \rightarrow i.e. x_1 = 84 \quad 1 \quad 8/5 \quad 0 \quad 3/5 \quad -1/5$$

$$R_2' = R_2 - \frac{1}{3} R_1'$$

$$100 - \frac{1}{3} + \frac{28}{8} \quad 1/3 - 1/3 \quad 1/3 - 8/5 + 1/3 \quad 1 - \frac{1}{3} \times 0 \quad 0 - \frac{3}{5} + \frac{1}{3} \quad \frac{1}{3} - \frac{1}{3} + \frac{1}{3}$$

$$\text{New } x_3 = 72 \quad 0 \quad -1/5 \quad 1 \quad -1/5 \quad 4/15 \quad 0$$

$$\text{Apply } R_3' = R_3 - \frac{2}{3} R_1' \\ \downarrow \quad \text{old } S_3 \quad \text{new } S_1 \text{ i.e. } (x_1)$$

$$200 - \frac{2}{3} \times 84 \quad 2/3 - 2/3 + 1 \quad 8/3 - \frac{2}{3} + 8/5 \quad 0 - \frac{2}{3} \times 0 \quad 0 - \frac{2}{3} \times \frac{3}{5} \quad -\frac{1}{3} - \frac{2}{3} + \frac{1}{2}$$

$$144 \quad 0 \quad 24/15 \quad 0 \quad -2/5 \quad -1/5$$

$$1 - \frac{2}{3} \times 0$$

1

$$S_3 \quad 144 \quad 0 \quad 24/15 \quad 0 \quad -2/5 \quad -1/5 \quad 1$$

Next iteration table

2	x_1	84	1	8/5	0	3/5	-1/5	0
4	x_3	72	0	-1/5	1	-1/5	4/15	0
0	S_3	144	0	24/15	0	-2/5	-4/15	0
		2	456	0	215	0	215	6/15

$$\therefore x_1 = 84, x_2 = 0, x_3 = 72 \quad \therefore z = 456$$

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$$\begin{aligned}
 \text{And } \text{Max } Z &= 2x_1 + 2x_2 + 4x_3 \\
 &= 2 \times 84 + 2 \times 0 + 4 \times 72 \\
 &= 168 + 0 + 288 \\
 &= \underline{\underline{456}}
 \end{aligned}$$

		x_1	x_2	x_3	S_1	S_2	S_3
0	S_1	240	2	3	1	1	0
0	S_2	300	1	1	3	0	1
0	S_3	300	1	3	1	0	0
Z		0	-2	-2	-4	0	0
0	S_1	140	$\frac{5}{3}$	$\frac{8}{3}$	0	1	$-\frac{1}{3}$
4	x_3	100	$\frac{1}{3}$	$\frac{1}{3}$	1	0	$\frac{1}{3}$
0	S_3	200	$\frac{2}{3}$	$\frac{8}{3}$	0	0	$-\frac{1}{3}$
Z		400	$-\frac{2}{3}$	$-\frac{2}{3}$	0	0	$\frac{4}{3}$
2	x_1	84	1	$\frac{8}{5}$	0	$\frac{3}{5}$	$-\frac{1}{5}$
4	x_3	72	0	$-\frac{1}{5}$	1	$-\frac{1}{5}$	$\frac{4}{5}$
0	S_3	144	0	$\frac{24}{5}$	0	$-\frac{2}{5}$	$-\frac{1}{5}$
Z		456	0	$\frac{2}{5}$	0	$\frac{2}{5}$	$\frac{6}{5}$

(Eq)

$$\text{Max } Z = x_1 - x_2 + 3x_3$$

Subject to UI constraint

$$x_1 + x_2 + x_3 \leq 10$$

$$2x_1 - x_3 \leq 2$$

$$2x_1 - 2x_2 + 3x_3 \leq 0$$

$$x_1, x_2, x_3 \geq 0$$

S.H?

$$x_1 + x_2 + x_3 + S_1 = 10$$

$$2x_1 + 0x_2 - x_3 + S_2 = 2$$

$$2x_1 + 2x_2 + 3x_3 + S_3 = 0$$

$$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$$

$$\therefore Z_{\max} = x_1 - x_2 + 3x_3 + 0S_1 + 0S_2 + 0S_3$$

	x_1	x_2	x_3	S_1	S_2	S_3	
R_1	0	S_1	10	1	1	1	$10x_1 = 10$
R_2	0	S_2	2	2	0	-1	$2x_1 - 1 = -2$
R_3	0	S_3	<u>0</u>	2	-2	3	$0x_3 = 0 \text{ min}$
	Z	0	-1	1	-3	0	0

$R'_1 = R_1 - R_3$	S_1	<u>10</u>	$\frac{1}{3}$	$\frac{5}{3}$	0	1	0	$\frac{10}{3} = \frac{10}{1} \times \frac{3}{5} = +ve$
$R'_2 = R_2 + R'_3$	S_2	2	$\frac{8}{3}$	$\frac{-2}{3}$	0	0	1	$\frac{-2}{3} = -ve$
$R'_3 = \frac{1}{3}R_3$	x_3	0	$\frac{2}{3}$	$\frac{-2}{3}$	1	0	0	$\frac{2}{3} = -ve$
	Z	0	1	-1	0	0	0	

$R''_1 = R'_1 + \frac{2}{3}R'_2$	S_1	6	$\frac{1}{5}$	1	0	$\frac{3}{5}$	0	$-\frac{1}{5}$
$R''_2 = R'_2 + 2R'_3$	S_2	4	$\frac{4}{5}$	0	0	$\frac{2}{5}$	1	$\frac{1}{5}$
$R''_3 = R'_3 + 2R''_1$	x_3	4	$\frac{4}{5}$	0	1	$\frac{2}{5}$	0	$\frac{1}{5}$

Z	6	$\frac{2}{5}$	0	0	$\frac{3}{5}$	0	$\frac{4}{5}$	
-----	---	---------------	---	---	---------------	---	---------------	--

Since all UI index numbers are true
the sol² is Optimum.

$$\therefore x_1 = 0, x_2 = 6, x_3 = 4$$

$$\therefore \text{Thus } Z_{\max} = x_1 - x_2 + 3x_3 = 0 - 6 + 3 \times 4 = 0 - 6 + 12 = 6$$

(Q) Solve the following LPP by the simplex method

$$\text{Minimize } Z = 2x_1 + 3x_2 + x_3$$

Subject to the following constraint

$$3x_1 + 2x_2 + x_3 \leq 3$$

$$2x_1 + x_2 + x_3 \leq 2$$

$$x_1, x_2, x_3 \geq 0.$$

Soln: As the given objective function is to minimize, it should be converted into maximization form.

$$M = Z_{\max} = -(Z_{\min})$$

$$Z_{\max} = -2x_1 - 3x_2 - x_3 + S_1 + S_2$$

RHS	x_1	x_2	x_3	S_1	S_2	
0	3	3	2	1	1	0
0	S_1	2	1	1	0	1
<hr/>						
Z	0	2	3	1	0	0

Index row values are positive.

$$\therefore x_1 = 0, x_2 = 0, x_3 = 0$$

The solution is optimum

$$\text{Hence } x_1, x_2, x_3 = 0$$

$$Z_{\min} = 0 \therefore Z_{\max} = 0$$

That is the minimum possible value is zero

subject to the constraint.

(eg) Solve the following LPP by simplex method.

$$\text{Min } Z = x_1 - 3x_2 + 2x_3$$

Subject to the constraint

$$3x_1 - x_2 + 3x_3 \leq 7$$

$$-2x_1 + 4x_2 \leq 12$$

$$-4x_1 + 3x_2 + 8x_3 \leq 10$$

$$x_1, x_2, x_3 \geq 0$$

Sol? Converting the objective function from minimization to maximization.

$$M = Z_{\max} = -(Z_{\min})$$

$$Z_{\max} = -x_1 + 3x_2 - 2x_3 + 0s_1 + 0s_2 + 0s_3$$

$$3x_1 - x_2 + 3x_3 + s_1 = 7$$

$$-2x_1 + 4x_2 + s_2 = 12$$

$$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$$

$$\therefore x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

	RHS	x_1	x_2	x_3	s_1	s_2	s_3	
R_1	s_1	7	3	-1	3	1	0	0
R_2	s_2	12	-2	4	0	0	1	0
R_3	s_3	10	-4	3	8	0	0	1
Z	0	1	-3	2	0	0	0	
$R'_1 = R_1 + R_2$	s_1	10	$\frac{s_2}{2}$	0	3	1	$\frac{1}{4}$	0
$R'_2 = \frac{1}{4}R_2$	x_2	3	$-\frac{1}{2}$	1	0	0	$\frac{1}{4}$	0
$R'_3 = R_3 - 3R_2$	s_3	1	$-\frac{s_2}{2}$	0	8	0	$-\frac{3}{4}$	1
Z	9	$-\frac{1}{2}$	0	2	0	$\frac{3}{4}$	0	
x_1	4	1	0	$\frac{6}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	0	
x_2	5	0	1	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	0	
s_3	11	0	0	$\frac{13}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0	
Z	11	0	0	$\frac{13}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0	

(11)

Hence $n_1 = 4$, $n_2 = 5$, $n_3 = 0$

$$\begin{aligned} M &= Z_{\max} = -n_1 + 3n_2 - 2n_3 \\ &= -4 + 3 \times 5 - 2 \times 0 \\ &= -4 + 15 \end{aligned}$$

$$Z_{\min} = \underline{\underline{11}}$$

$$\therefore M = -(Z_{\min}) = Z_{\max}$$

$$Z_{\min} = \underline{\underline{-11}}$$

Tie breaking in the simplex method

Problem of degeneracy (Tie for minimum ratio) in LPP

Sometimes the key element is not uniquely determined or at the very first iteration the value of one or more basic variables in the column become equal to zero, this causes the problem of degeneracy.

If the minimum ratio is zero for two or more basic variables, degeneracy may lead the simplex routine to cycle indefinitely. The solution obtained in one iteration may repeat again after few iterations, therefore no optimum solution may be obtained under such circumstances.

Method to Resolve Tie (Degeneracy)

Step 1: Pick up the rows for which the minimum non negative ratio is same (i.e. tied)
for eg: let us say for the 1st & 3rd row.

Step 2: Rearrange the columns of the usual simplex table so that columns forming the original unit matrix or (identity matrix) comes first in the proper order.

Step 3: Then find the minimum of the ratios

i.e.
$$\frac{\text{Elements of 1st column of identity matrix}}{\text{corresponding elements of key column}}$$

only for the rows for which minimum ratio is unique. That is for the rows first & third as picked up in step(1).

- (i) If this minimum is attained for 3rd row (say) then this row will determine the key element by intersecting the key column.
- (ii) If this minimum is also not unique, then go to the next step.

Step 4: In this step compute the minimum of U_i ratio:

$$\left[\frac{\text{Elements of 2}^{\text{nd}} \text{ column of identity matrix}}{\text{corresponding elements of key column}} \right]$$

only for the rows for which minimum was not unique in Step 3.

- (i) If U_i min ratio is unique for U_i first row (say) then this row will determine U_i key element by intersecting U_i key column.
- (ii) If this min is still not unique then go to the next step.

Step 5: In this step compute U_i min of U_i ratio

$$\left[\frac{\text{Elements of 3}^{\text{rd}} \text{ column of identity matrix}}{\text{corresponding elements of key column}} \right]$$

only for the rows for which min ratio is not unique in step(4).

- (i) If this min ratio is unique for U_i third row (say) then this row will determine U_i key element by intersecting the key column.
- (ii) If this min is still not unique, then go on repeating the above procedure till the unique min ratio is obtained to resolve the degeneracy. After the resolution of this tie, usual simplex method is applied to obtain the optimum solution.

(eg) Solve the following LPP by simplex method
 S.T.C. $4x_1 + 3x_2 \leq 12$ $\text{Max } Z = 2x_1 + x_2$
 $4x_1 + x_2 \leq 8$
 $4x_1 - x_2 \leq 8$
 $x_1, x_2 \geq 0$.

S.M? $4x_1 + 3x_2 + s_1 = 12$

$4x_1 + x_2 + s_2 = 8$

$4x_1 - x_2 + s_3 = 8$

$x_1, x_2, s_1, s_2, s_3 \geq 0$.

& $\text{Max } Z = 2x_1 + x_2 + 0s_1 + 0s_2 + 0s_3$

	2	1	0	0	0
RHS	x_1	x_2	s_1	s_2	s_3

$R_1 \quad 0 \ s_1 \ 12 \quad | \ 4 \ 3 \ 1 \ 0 \ 0 \quad 12|_4 = 3$

$R_2 \quad 0 \ s_2 \ 8 \quad | \ 4 \ 1 \ 0 \ 1 \ 0 \quad 8|_4 = 2 \}$

$R_3 \quad 0 \ s_3 \ 8 \quad | \ 4 \ -1 \ 0 \ 0 \ 1 \quad 8|_4 = 2 \}$

$Z \ 0 \ | \ -2 \ -1 \ 0 \ 0 \ 0$

$s_1 \ s_2 \ s_3 \ x_1 \ x_2$

$s_1/x_1, s_2/x_2, \dots$

$R'_1 = R_1 - 4R_3 \quad 0 \ s_1 \ 12 \quad | \ 1 \ 0 \ 0 \quad 4 \ 3$

$R'_2 \quad 0 \ s_2 \ 8 \quad | \ 0 \ 1 \ 0 \quad 4 \ 1$

$R'_3 \quad 0 \ s_3 \ 8 \quad | \ 0 \ 0 \ 1 \quad 4 \ -1$

$Z \ 0 \ 0 \ 0 \ 0 \ | \ -2 \ -1$

$s_1 \ 4 \ 1 \ 0 \ -1 \ 0 \ | \ 4$

$s_2 \ 0 \ 0 \ 1 \ -1 \ 0 \ | \ 2$

$x_1 \ 2 \ 0 \ 0 \ 1/4 \ 1 \ | \ -1/4$

$Z \ 4 \ 0 \ 0 \ 1/4 \ 0 \ | \ -3/2$

$s_1 \ 4 \ 1 \ -2 \ 1 \ 0 \ 0$

$x_2 \ 0 \ 0 \ 1/2 \ -1/2 \ 0 \ 1$

$x_1 \ 2 \ 0 \ 1/8 \ 1/8 \ 1 \ 0$

$Z \ 4 \ 0 \ 3/4 \ -1/4 \ 0 \ 0$

$s_3 \ 4 \ 1 \ -2 \ 1 \ 0 \ 0$

$x_2 \ 2 \ 1/2 \ -1/2 \ 0 \ 0 \ 1$

$x_1 \ 3/2 \ -1/8 \ 3/8 \ 0 \ 1 \ 0$

$0/4 \} \text{ Tie occurs}$

$0/4 \} \quad 1/4 = 0.25$

$0/4 = 0$

$4|_4 = 1$

$0|_2 = 0$

$2 + -1/4 = \text{re.}$

$4|_1 = 4$

$0| -1/2 = \text{re.}$

$2/1/8 = 2 \times 8 = 16$

Since all index numbers are positive the solution is optimal

$$f(x) = 3x_1 + 2x_2$$

$$Z_{\max} = 5$$

$$Z_{\max} = 2x_1 + x_2$$

$$= 2(3) + 2 = 5$$

Eg Solve the following LPP by simplex method

$$\text{Max } Z = 3x_1 + 2x_2$$

Subject to the constraint

$$4x_1 + 3x_2 \leq 12$$

$$4x_1 + x_2 \leq 8$$

$$4x_1 - x_2 \leq 8$$

$$x_1, x_2 \geq 0$$

Sol: After solving using the simplex method we will get
 $x_1 = 3/2, x_2 = 2 \quad Z_{\max} = 17/2$

Eg: Use the simplex method to solve the following LPP

$$\text{Maximize } Z = x_1 + x_2$$

Subject to the constraint

$$x_1 + 5x_2 \leq 5$$

$$2x_1 + x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Sol: After solving using the simplex method we will get the answer as

$$x_1 = 5/3, x_2 = 2/3$$

$$\therefore Z_{\max} = 7/3$$

Solution to LPP by two-phase method

Steps involved in Two-phase simplex method

Phase-I: In this phase, the simplex method is applied to a specially constructed auxiliary linear programming problem leading to a final simplex table containing a basic feasible solution to the original problem.

Step 1: Assign a cost = -1 to each artificial variable and a cost = 0 to all other variables (in place of their original cost) in the objective function.

Step 2: Construct an auxiliary LPP in which the new objective function Z^* is to be maximized subject to the given set of constraints.

Step 3: Solve the auxiliary problem by simplex method until either of the following three cases arise.

(i) $\text{Max } Z^* < 0$ & at least one artificial variable appears in the optimum basis at a positive level. In this case the given problem does not possess any feasible solution.

(ii) $\text{Max } Z^* = 0$ & at least one artificial variable appears in the optimum basis at zero level. In this case proceed to phase-II.

(iii) $\text{Max } Z^* = 0$ & no artificial variable appears in the optimum basis. In this case also proceed to phase-II.

Phase-II: In this phase assign the actual cost to the variable in the objective function & zero cost to every artificial variable that appears in the basis at the zero level. This new objective function is then maximized by simplex method subject to the given constraints i.e. simplex method is applied to the

modified simplex table obtained at the end of phase-I until an optimum basic feasible solution (if exists) has been attained. The artificial variable which are non-basic at the end of phase-I are removed.

(Q) Solve the following LPP using two-phase method.

$$\text{Maximize } Z = 5x_1 - 4x_2 + 3x_3$$

Subject to the constraint

$$2x_1 + x_2 - 6x_3 = 20$$

$$6x_1 + 5x_2 + 10x_3 \leq 76$$

$$8x_1 - 3x_2 + 6x_3 \leq 50$$

$$x_1, x_2, x_3 \geq 0$$

Sol: Phase-I

$$\text{Max } Z = 0x_1 + 0x_2 + 0x_3 - 1a_1 + 0S_1 + 0S_2$$

$$2x_1 + x_2 - 6x_3 + a_1 = 20$$

$$6x_1 + 5x_2 + 10x_3 + S_1 = 76$$

$$8x_1 - 3x_2 + 6x_3 + S_2 = 50$$

$$x_1, x_2, x_3, a_1, S_1, S_2 \geq 0$$

$$0 \quad 0 \quad 0 \quad -1 \quad 0 \quad 0$$

$$x_1 \quad x_2 \quad x_3 \quad a_1 \quad S_1 \quad S_2$$

$$-1 \ a_1 \ 20 \ 2 \ 1 \ -6 \ 1 \ 0 \ 0$$

$$0 \ S_1 \ 76 \ 6 \ 5 \ 10 \ 0 \ 1 \ 0$$

$$0 \ S_2 \ 50 \ 8 \ -3 \ 6 \ 0 \ 0 \ -1$$

$$\underline{\underline{Z \ -20 \ -2 \ -1 \ 6 \ 0 \ 0 \ 0}}$$

$$-1 \ a_1 \ 15/2 \ 0 \ 7/4 \ -15/2 \ 1 \ 0 \ -1/4$$

$$0 \ S_1 \ 77/2 \ 0 \ 29/4 \ 11/2 \ 0 \ 1 \ -3/4$$

$$0 \ x_1 \ 25/4 \ 1 \ -3/8 \ 3/4 \ 0 \ 0 \ 1/8$$

$$\underline{\underline{Z \ -15/2 \ 0 \ -7/4 \ 15/2 \ 0 \ 0 \ 1/4}}$$

$$x_2 \ 30/7 \ 0 \ 1 \ -30/7 \ 4/7 \ 0 \ -1/7$$

$$S_1 \ 52/7 \ 0 \ 0 \ 258/7 \ -29/7 \ 1 \ 2/7$$

$$x_1 \ 55/7 \ 1 \ 0 \ -6/7 \ 3/4 \ 0 \ 1/14$$

$$\underline{\underline{Z \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0}}$$

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Since all the index row numbers are ≥ 0 an optimum solution to the auxiliary LPP has been obtained. Further the value of $\max Z = 0$ with no artificial variables in the basic solution, proceed to phase-II.

Phase-II: In phase-II consider the final simplex table of phase-I. Consider actual costs associated with the original variables & also delete variable column a_1 from the final table of phase-I. Then the objective function becomes

$$\text{Max } Z = 5x_1 - 4x_2 + 3x_3 + 0S_1 + 0S_2$$

	5	-4	3	0	0
	x_1	x_2	x_3	S_1	S_2
-4	x_2	$30/7$	0	1	$-30/7$
0	S_1	$52/7$	0	0	$256/7$
5	x_1	$55/7$	1	0	$-6/7$
<hr/>					
Z		$155/7$	0	0	$69/7$
					$13/14$

$$\therefore x_1 = \frac{55}{7}, x_2 = \frac{30}{7}, x_3 = 0$$

$$Z_{\max} = \frac{155}{7}$$

(Q) Solve the following LPP by the using two phase simplex method.

$$\text{Minimize } Z = 3x_1 - x_2$$

Subject to the constraint

$$2x_1 + x_2 \geq 2$$

$$x_1 + 3x_2 \leq 2$$

$$x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Soln.Phase-I

$$2x_1 + x_2 - s_1 + a_1 = 2$$

$$x_1 + 3x_2 + s_2 = 2$$

$$x_2 + s_3 = 4$$

$$x_1, x_2, s_1, s_2, s_3, a_1 \geq 0$$

$$\text{Max } Z = 0x_1 - 0x_2 + 0s_1 + 0s_2 + 0s_3 - 1a_1$$

	x_1	x_2	s_1	s_2	s_3	a_1	
-1	a_1	2	2	1	-1	0	0
0	s_2	2	1	3	0	1	1
0	s_3	4	0	1	0	0	1
Z	-2	-2	-1	1	0	0	0
x_1	1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$
s_2	1	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$-\frac{1}{2}$
s_3	4	0	1	0	0	0	0
Z	0	0	0	0	0	0	1

Since all the index row numbers are ≥ 0 , an optimum solution to the auxiliary LPP is obtained. Further the value of $Z=0$ with no artificial variable in the basic solution proceed to Phase-II.

Phase-II

objective function

$$Z_{\max} = 3x_1 - x_2 + 0s_1 + 0s_2 + 0s_3$$

	x_1	x_2	s_1	s_2	s_3	
3	x_1	1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0
0	s_2	1	0	$\frac{1}{2}$	$\frac{1}{2}$	1
0	s_3	4	0	1	0	0
Z	3	0	$\frac{1}{2}$	$-\frac{3}{2}$	0	0
x_1	2	1	3	0	1	0
s_1	2	0	5	1	2	0
s_3	4	0	1	0	0	1
Z	6	0	10	0	3	0

Since all the index low numbers are ≥ 0
an optimum basic feasible solution is obtained.

$$\therefore x_1 = 2, x_2 = 0$$

$$Z_{\max} = 6$$

(Q) Use two-phase simplex method to solve the following LPP

$$\text{Maximize } Z = -x_1 + 2x_2 + 3x_3$$

subject to the constraint

$$-2x_1 + x_2 + 3x_3 = 2$$

$$2x_1 + 3x_2 + 4x_3 = 1$$

$$x_1, x_2, x_3 \geq 0$$

Sol:

$$-2x_1 + x_2 + 3x_3 + a_1 = 2$$

$$2x_1 + 3x_2 + 4x_3 + a_2 = 1$$

$$x_1, x_2, x_3, a_1, a_2 \geq 0$$

Phase-I

The auxiliary LPP is

$$Z_{\max} = 0x_1 + 0x_2 + 0x_3 - 1a_1 - 1a_2$$

	x_1	x_2	x_3	a_1	a_2	
-1	a_1	2	-2	1	3	1
-1	a_2	1	2	3	4	0
	Z	-3	0	=4	-7	0
	a_1	$\frac{1}{4}$	$-\frac{7}{2}$	$-\frac{9}{4}$	0	$\frac{1}{4}$
	x_3	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	0
	Z	$-\frac{9}{4}$	$\frac{7}{2}$	$\frac{5}{4}$	0	$\frac{1}{4}$

Since all the net evaluation or index low numbers are ≥ 0 an optimum basic feasible solution to the auxiliary LPP has been attained.
But the value of $\max Z$ is $-ve$ & the artificial variable a_1 appears in the basic solution at a positive level. Hence the original problem does not possess any feasible soln. Therefore there is no need to pass to phase-II.

Q8 Solve the following LPP using two-phase simplex method

$$\text{Minimize } Z = \frac{15}{2}x_1 - 3x_2$$

Subject to the constraint

$$3x_1 - x_2 - x_3 \geq 3$$

$$x_1 - x_2 + x_3 \geq 2$$

$$\text{s.t. } x_1, x_2, x_3 \geq 0$$

Sol:

$$Z_{\text{min}} = -\frac{15}{2}x_1 + 3x_2$$

$$3x_1 - x_2 - x_3 - s_1 + a_1 = 3$$

$$x_1 - x_2 + x_3 - s_2 + a_2 = 2$$

$$x_1, x_2, x_3, s_1, s_2, a_1, a_2 \geq 0$$

$$\begin{array}{ccccccc} \text{Max } Z' & = & 0x_1 & + 0x_2 & + 0x_3 & + 0s_1 & + 0s_2 - 1a_1 - 1a_2 \\ & & 0 & 0 & 0 & 0 & -1 -1 \\ & & x_1 & x_2 & x_3 & s_1 & s_2 & a_1 & a_2 \end{array}$$

$$\begin{array}{ccccccccc} -1 & a_1 & 3 & 3 & -1 & -1 & -1 & 0 & 1 & 0 \end{array}$$

$$\begin{array}{ccccccccc} -1 & a_2 & 2 & 1 & -1 & 1 & 0 & -1 & 0 & 1 \end{array}$$

$$\begin{array}{ccccccccc} \hline Z & -5 & -4 & 2 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}$$

$$\begin{array}{ccccccccc} x_1 & 1 & 1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \end{array}$$

$$\begin{array}{ccccccccc} a_2 & 1 & 0 & -\frac{4}{3} & \frac{4}{3} & \frac{1}{3} & -1 & \frac{1}{3} & 1 \end{array}$$

$$\begin{array}{ccccccccc} \hline Z & -1 & 0 & \frac{2}{3} & -\frac{4}{3} & -\frac{1}{3} & 1 & \frac{4}{3} & 0 \end{array}$$

$$\begin{array}{ccccccccc} x_1 & \frac{5}{4} & 1 & -\frac{1}{2} & 0 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{array}$$

$$\begin{array}{ccccccccc} x_3 & \frac{3}{4} & 0 & -\frac{1}{2} & 1 & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & \frac{3}{4} \end{array}$$

$$\begin{array}{ccccccccc} \hline Z & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}$$

Phase-II: Max $Z' = -\frac{15}{2}x_1 + 3x_2 + 0s_1 + 0s_2$

$$\begin{array}{ccccccccc} -\frac{15}{2}x_1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$\begin{array}{ccccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & & & & \end{array}$$

$$\begin{array}{ccccccccc} -\frac{15}{2}x_1 & \frac{5}{4} & 1 & -\frac{1}{2} & 0 & -\frac{1}{4} & -\frac{1}{4} & & \end{array}$$

$$\begin{array}{ccccccccc} 3x_3 & \frac{3}{4} & 0 & -\frac{1}{2} & 1 & \frac{1}{4} & -\frac{3}{4} & & \end{array}$$

$$\begin{array}{ccccccccc} \hline Z & -\frac{25}{8} & 0 & \frac{3}{4} & 0 & \frac{15}{8} & \frac{15}{8} & & \end{array}$$

An optimum basic feasible solution has been obtained.

$$x_1 = \frac{5}{4}, \quad x_2 = 0, \quad x_3 = \frac{3}{4}$$

$$Z_{\max} = -\frac{75}{8}$$

$$\boxed{Z_{\min} = \frac{75}{8}}$$

(Q) Solve the following LPP using two phase method

$$\text{Max } Z = 5x_1 + 8x_2$$

Subject to the constraint

$$3x_1 + 2x_2 \geq 3$$

$$x_1 + 4x_2 \geq 4$$

$$x_1 + x_2 \leq 5$$

$$\& x_1, x_2 \geq 0.$$

Soln After solving we will get answer as

$$x_1 = 0, \quad x_2 = 5, \quad \text{Max } Z = 40.$$

Big M method

Computational steps of Big-M method is as follows:

- Step 1: Express the problem in the Standard LPP form
- Step 2: If the constraints are \leq add slack variables only on the left hand side of the constraint eqn.
- Step 3: If the constraints are \geq subtract the surplus variable & add the artificial variable on the left hand side of the constraints.
- Step 4: If the constraints are $=$ (equal to) add artificial variables only on the left hand side of the constraint. The main purpose of introducing the artificial variables is to get the initial basic feasible solution (IBFS).
- Step 5: The coefficients of the artificial variables in the objective row is $-M$, where M is a huge value unspecified but sufficiently large.
- Step 6: In the initial variable column write only the slack & artificial variables.
- Step 7: Then follow the steps involved in the simplex method.

(Eg)

Solve the following LPP by Big-M method

$$Z_{\max} = 3x_1 - x_2$$

Subject to the constraint

$$2x_1 + x_2 \geq 2$$

$$x_1 + 3x_2 \leq 3$$

$$x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Ex 1

better M.p.t

$$\text{Max } Z = 3x_1 - x_2 + 0s_1 + 0s_2 + 0s_3 - Ma_1$$

$$2x_1 + x_2 - s_1 + a_1 = 2$$

$$x_1 + 3x_2 + s_2 = 3$$

$$x_2 + s_3 = 4$$

$$x_1, x_2, s_1, s_2, s_3, a_1 \geq 0$$

	3	-1	0	0	0	-M
	x_1	x_2	s_1	s_2	s_3	a_1

$$0 \quad s_2 \quad 3 \quad | \quad 1 \quad 3 \quad 0 \quad 1 \quad 0 \quad 0 \quad 3 \quad 3s_1 = 3$$

$$0 \quad s_3 \quad 4 \quad | \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 4 \quad 4s_2 = 0$$

$$-M \quad a_1 \quad 2 \quad 2 \quad | \quad 1 \quad -1 \quad 0 \quad 0 \quad 1 \quad 2 \quad 2a_1 = 1$$

$$Z \quad -2m \quad -2m \quad -m+1 \quad m \quad 0 \quad 0 \quad 0$$

$$s_2 \quad 2 \quad 0 \quad 5/2 \quad | \quad 1/2 \quad 1 \quad 0 \quad -$$

$$s_3 \quad 4 \quad 0 \quad 1 \quad | \quad 0 \quad 0 \quad 1 \quad -$$

$$x_1 \quad 1 \quad 1 \quad 1/2 \quad | \quad -1/2 \quad 0 \quad 0 \quad -$$

$$Z \quad 3 \quad 0 \quad 5/2 \quad | \quad -3/2 \quad 0 \quad 0 \quad -$$

$$s_1 \quad 4 \quad 0 \quad 5 \quad | \quad 1 \quad 2 \quad 0 \quad -$$

$$s_3 \quad 4 \quad 0 \quad 1 \quad | \quad 0 \quad 0 \quad 1 \quad -$$

$$x_1 \quad 3 \quad 1 \quad 3 \quad | \quad 0 \quad 1 \quad 0 \quad -$$

$$Z \quad 9 \quad 0 \quad 10 \quad | \quad 0 \quad 3 \quad 0 \quad -$$

Since all index row numbers are positive the solution is optimum.

$$\therefore x_1 = 3, x_2 = 0$$

$$Z_{\max} = 9$$

$$Z_{\max} = 3x_1 - x_2$$

$$= 3 \times 3 - 0$$

$$= 9 - 0$$

$$\underline{Z_{\max} = 9}$$

Ex

Solve the following LPP by Big-M method

Minimize $Z = 2x_1 + x_2$
Subject to the constraint

$$3x_1 + x_2 = 3$$

$$4x_1 + 3x_2 \geq 6$$

$$x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Soln: Converting the objective function from minimization form to maximization.

i.e. Max Z is declared

$$\text{i.e. } \text{Max } Z = -\min Z = \max Z$$

$$\therefore \max Z' = -2x_1 - x_2$$

Now transforming the inequality constraints into eqns

$$3x_1 + x_2 + a_1 = 3$$

$$4x_1 + 3x_2 - s_1 + a_2 = 6$$

$$x_1 + 2x_2 + s_2 = 4$$

and $\max Z' = -2x_1 - x_2 + 0s_1 + 0s_2 - Ma_1 - Ma_2$

	-2	-1	0	0	$-M$	$-M$
x_1	a_1	a_2	s_1	s_2	a_1	a_2

0	s_2	4	1	2	0	1	0	0
$-M$	a_1	3	3	1	0	0	1	0
$-M$	a_2	6	4	3	-1	0	0	1
Z'		$-9M$	$-7M+2$	$-4M+1$	M	0	0	0
s_2		3	0	$5/3$	0	1	$-1/3$	0
x_1		1	1	$1/3$	0	0	$1/3$	0
a_2		2	0	$5/3$	-1	0	$-4/3$	1

Z'	$-2m-2$	0	$\frac{-5m+1}{3}$	m	0	$\frac{(7m-2)}{3}$	0
s_2	1	0	0	1	1	1	-1
x_1	$3/5$	1	0	$1/5$	0	$3/5$	$-1/5$
x_2	$6/5$	0	1	$-3/5$	0	$-4/3$	$3/5$

Z'	$-12/5$	0	0	$1/5$	0	$15m-6$	$5m-1$
------	---------	---	---	-------	---	---------	--------

$$x_1 = \frac{3}{5}, \quad x_2 = \frac{6}{5} \quad \max z = -2\left(\frac{3}{5}\right) - \left(\frac{6}{5}\right)$$

$$\max z = -\frac{6+6}{5} = -\frac{12}{5}$$

$$\min z = \frac{12}{5}$$

(Eq) Solve the following LPP using Big-M method

$$\max z = 2x_1 + 3x_2 + 10x_3$$

$$\text{Subject to the constraint } x_1 + 2x_3 = 0$$

$$x_2 + x_3 = 1$$

$$x_1, x_2, x_3 \geq 0$$

Sol:
Sap:

$$x_1 + 2x_3 + a_1 = 0$$

$$x_2 + x_3 + a_2 = 1$$

$$x_1, x_2, x_3, a_1, a_2 \geq 0.$$

$$\max z = 2x_1 + 3x_2 + 10x_3 - Ma_1 - Ma_2$$

	x_1	x_2	x_3	a_1	a_2	
$-Ma_1$	0	1	0	2	1	0
$-Ma_2$	1	0	1	1	0	1
Z	-M	-M-2	-M-3	$-3M-10$	0	0
x_3	0	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0
a_2	1	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	1
Z	-M	$\frac{m+6}{2}$	-M-3	0	$\frac{3m+10}{2}$	0
x_3	0	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0
x_2	1	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	1
Z	3	$\frac{3}{2}$	0	0	$\frac{2m+7}{2}$	$m+3$

$$\therefore x_1 = 0, x_2 = 1, x_3 = 0.$$

$$Z_{\max} = 3$$

$$\max z = 2x_0 + 3x_1 + 10x_0.$$

Z_{max} = 3

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Solve the following LPP by Big-M method

Q8

$$\text{Max } Z = 3x_1 + 5x_2$$

Subject to the constraint

$$x_1 + x_3 = 4$$

$$x_2 + x_4 = 6$$

$$3x_1 + 2x_2 + x_5 = 12$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

SOL:

$$\text{Max } Z = 3x_1 + 5x_2 + 0x_3 + 0x_4 + 0x_5 - Ma_1 - Ma_2 - Ma_3$$

$$x_1 + x_3 + a_1 = 4$$

$$x_2 + x_4 + a_2 = 6$$

$$3x_1 + 2x_2 + x_5 + a_3 = 12$$

$$x_1, x_2, x_3, x_4, x_5, a_1, a_2, a_3 \geq 0$$

	x_1	x_2	x_3	x_4	x_5	a_1	a_2	a_3	
$-m a_1$	4	1	0	1	0	0	1	0	0
$-m a_2$	6	0	1	0	1	0	0	1	0
$-m a_3$	12	3	2	0	0	1	0	0	1
Z	$-2x_1 - 4x_2 - 3x_3 - 5x_4 - m$	$-4m - 3$	$-3m - 5$	$-m$	$-m$	$-m$	0	0	0

* Note: Since both are artificial Variable any one can be

	x_1	x_2	x_3	x_4	x_5	a_1	a_2	a_3	
3 x_1	4	1	0	1	0	0	1	0	$4/0 = 4$
$-m a_2$	6	0	1	0	1	0	0	1	$6/1 = 6$
$-m a_3$	0	0	2	-3	0	1	-3	0	$0/2 = 0$
Z	$-6m + 12$	0	$-3m - 5$	$3m + 5$	$-m$	$-m$	$4m + 3$	0	0
x_1	4	1	0	1	0	0	1	0	0
a_2	6	0	0	$3/2$	1	$-1/2$	$3/2$	1	$-1/2$
x_2	0	0	1	$-3/2$	0	$1/2$	$-3/2$	0	$1/2$
Z	$-6m + 12$	0	0	$\frac{-3m - 9}{2}$	$-m$	$\frac{m + 5}{2}$	$\frac{-m - 9}{2}$	0	$\frac{3m + 5}{2}$

Note: Give preference to artificial variable to leave the solution.

	x_1	x_2	x_3	x_4	x_5	a_1	a_2	a_3
x_1	0	1	0	0	$-2/3$	$1/3$	0	$-2/3$
x_3	4	0	0	1	$2/3$	$-1/3$	1	$2/3$
x_2	6	0	1	0	1	0	0	1
Z	30	0	0	0	3	1	M	$m+3$
								$m+1$

$$\therefore x_1 = 0, x_2 = 6, x_3 = 4, x_4 = 0, x_5 = 0.$$

$$\text{Max } Z = 30$$

(eg) Solve the following LPP by Big M-method

$$\text{Maximize } Z = 3x_1 + 2x_2$$

subject to the constraint

$$2x_1 + x_2 \leq 2$$

$$3x_1 + 4x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

Sol?

$$2x_1 + x_2 + S_1 = 2$$

$$3x_1 + 4x_2 - S_2 + a_1 = 12$$

$$x_1, x_2, S_1, S_2, a_1 \geq 0$$

$$\text{Max } Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 - Ma_1$$

	3	2	0	0	-m
	x_1	x_2	S_1	S_2	a_1
$0 S_1$	2	2	1	1	0
$-m a_1$	12	3	4	0	-1
Z	$-12m$	$-3m-3$	$-4m-2$	0	M
x_2	2	2	1	1	0
a_1	4	-5	0	-4	-1
Z	$-4m+4$	$5m+1$	0	$4m+2$	M

Here all index row numbers are ~~not~~ positive & the value of Z is in terms of m . Thus the given LPP does not possess an optimum basic feasible solution & there exists a pseudo-optimum solution.

Simplex method - 2, Duality theory

The Essence of Duality Theory (concept of Duality)

In LPP duality implies that each problem can be analyzed in two different ways but having equivalent solutions. Each LP problem stated in the original form is associated with another LPP called dual LPP or its dual. The two problems are replica of each other. Dual of the primal problem is unique. The simplex rule is such that if the primal is solved it is equivalent to solving the dual.

Key Relationships Between Primal & Dual problems

Weak duality property: If x is a feasible solution for the primal then y is a feasible solution for the dual problem then, $Cx \leq yb$, this inequality must hold for any pair of feasible solutions of primal & dual.

Strong duality property: If x^* is an optimal solution for the primal & y^* is an optimal solution for the dual problem, then $Cx^* = y^* b$.

Complementary solutions property: At each iteration the simplex method simultaneously identifies a solution x for the primal problem & a complementary solution y for the dual problem. In $Cx = yb$ if x is optimal for primal problem then y is not feasible for dual.

Complementary optimal solutions property: At the final iteration the simplex method simultaneously identifies an optimal solution x^* for the primal problem & a complementary optimal solution y^* for the dual problem. The relationship can be expressed as $Cx^* = y^* b$.

Primal Dual Relationship:

Let the primal problem be

$$Z_{\text{max}} = C_1 x_1 + C_2 x_2 + \dots + C_n x_n$$

Subject to

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \leq b_1$$

$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n \leq b_2$$

⋮

$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n \leq b_m$$

$$x_1, x_2, \dots, x_n \geq 0$$

Dual of the problem is defined as (let w_1, w_2, w_3 be the dual variables)

$$\text{Max } Z^* = b_1 w_1 + b_2 w_2 + \dots + b_m w_m$$

Subject to

$$a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m \geq C_1$$

$$a_{12} w_1 + a_{22} w_2 + \dots + a_{m2} w_m \geq C_2$$

⋮

⋮

Characteristics of the Dual problem

Duality in linear programming has the following main characteristics.

- i) Dual of the dual LP problem is primal.
- ii) If either the primal or dual of the problem has an optimal solution, then the other one also will have same.
- iii) If any of the two problems has only an infeasible solution, then the value of the objective function of the other is unbounded.
- iv) The value of the objective function for any feasible solution of the primal is less than the value of the objective function for any feasible solution of the dual.
- v) If either the primal or the dual has an unbounded objective function value then the solution to the other problem is infeasible.
- vi) If the primal has a feasible solution but the dual does not have, then the primal will not have finite optimal solution & vice-versa.

Advantages of Duality

- (i) Yields a number of useful theorems.
- (ii) Solves the dual problem which is often easier than the primal problem.

- III) Indicates that fairly close relationship exists between linear programming & duality.
- IV) Economic interpretation of the dual helps the management in making future decisions.
- V) Computational procedure can be considered redundant.
- VI) Duality can be used to solve an LP problem by the simplex method in which the initial solution is infeasible.

(Eg)

Write the dual of the following LPP

$$\text{Max } Z = x_1 + 2x_2 + x_3$$

Subject to

$$2x_1 + x_2 - x_3 \leq 2$$

$$-2x_1 + x_2 - 5x_3 \geq -6$$

$$4x_1 + x_2 + x_3 \leq 6$$

$$x_1, x_2, x_3 \geq 0$$

Sol:

$$\text{Max } Z = x_1 + 2x_2 + x_3$$

Subject to

$$2x_1 + x_2 - x_3 \leq 2$$

Note: put the 2nd constraint in standard form

$$2x_1 - x_2 + 5x_3 \leq 6$$

$$4x_1 + x_2 + x_3 \leq 6 \quad \text{s.t. } x_1, x_2, x_3 \geq 0$$

Dual: Let w_1, w_2, w_3 be the dual variables

(Q) Write dual of the following LPP

$$\text{Max } Z = 3x_1 - x_2 + x_3$$

$$\text{Subject to } 4x_1 - x_2 \leq 8, \quad 8x_1 + x_2 + 3x_3 \geq 12,$$

$$5x_1 - 6x_3 \leq 13 \quad \text{and} \quad x_1, x_2, x_3 \geq 0$$

Sol?

$$\text{Max } Z = 3x_1 - x_2 + x_3$$

$$4x_1 - x_2 \leq 8$$

Put the 2nd constraint in std form (\leq form)

$$\text{i.e. } -8x_1 - x_2 - 3x_3 \leq -12$$

$$5x_1 - 6x_3 \leq 13$$

Dual: $\text{Min } Z^* = 8w_1 - 12w_2 + 13w_3$

Subject to

$$4w_1 - 8w_2 + 5w_3 \geq 3$$

$$-w_1 - w_2 - 3w_3 \geq -1$$

$$\text{& } 0w_1 - 3w_2 - 6w_3 \geq 1 \quad \text{& } w_1, w_2, w_3 \geq 0$$

(Q)

Write the dual of the following LPP

$$\text{Min } Z = 0.4x_1 + 0.5x_2$$

$$\text{Subject to } 0.3x_1 + 0.1x_2 \geq 2.7$$

$$0.5x_1 + 0.5x_2 \geq 6$$

$$0.6x_1 + 0.4x_2 \geq 6 \quad \text{& } x_1, x_2 \geq 0$$

Sol?

$$\text{Max } Z^* = 2.7w_1 + 6w_2 + 6w_3$$

Subject to

$$0.3w_1 + 0.5w_2 + 0.6w_3 \leq 0.4$$

$$0.1w_1 + 0.5w_2 + 0.4w_3 \leq 0.5$$

$w_1 \geq 0$, w_2 is unconstrained as it is a RHS

(Eq) write the dual of the following LPP

$$\text{Max } Z = 40x_1 + 35x_2$$

subject to

$$2x_1 + 3x_2 \leq 60$$

$$4x_1 + 3x_2 \leq 96$$

$$x_1, x_2 \geq 0.$$

Soln: Dual

$$\text{Min } Z^2 = 60w_1 + 96w_2$$

$$2w_1 + 4w_2 \geq 40$$

$$3w_1 + 3w_2 \geq 35$$

$$w_1, w_2 \geq 0.$$

Dual of above dual

$$\text{Max } Z = 40x_1 + 35x_2$$

subject to

$$2x_1 + 3x_2 \leq 60$$

$$4x_1 + 3x_2 \leq 96$$

$$x_1, x_2 \geq 0.$$

(Eq) write the dual of the following LPP

$$\text{minimize } Z = 40x_1 + 24x_2$$

subject to

$$20x_1 + 50x_2 \geq 4800$$

$$80x_1 + 50x_2 \geq 7200$$

$$x_1, x_2 \geq 0.$$

Soln: $\text{Max } Z^2 = 4800w_1 + 7200w_2$

subject to

$$20w_1 + 80w_2 \leq 40$$

(4)

Eg write the dual of the following LPP.

$$\text{Maximize } Z = 3x_1 + 4x_2 + 7x_3$$

subject to

$$x_1 + x_2 + x_3 \leq 10$$

$$4x_1 - x_2 - x_3 \geq 15$$

$$x_1 + x_2 + x_3 = 7$$

$x_1, x_2 \geq 0$, x_3 is unrestricted in sign

$$\rightarrow \begin{array}{l} \text{Maximize } \\ -4x_1 + x_2 + x_3 \leq -15 \end{array}$$

Take $x_3 = (x_4 - x_5)$ where $x_4 \geq 0$ and $x_5 \geq 0$.

$$x_1 + x_2 + x_3 = 7$$

$$x_1 + x_2 + x_3 \leq 7 \quad x_1 + x_2 + x_3 \geq 7$$

$$-x_1 - x_2 - x_3 \leq -7$$

Revised primal

$$\text{Maximize } Z = 3x_1 + 4x_2 + 7x_4 - 7x_5$$

subject to

$$x_1 + x_2 + x_4 - x_5 \leq 10 \quad w_1$$

$$-4x_1 + x_2 + x_3 \leq -15 \quad w_2$$

$$x_1 + x_2 + x_4 - x_5 \leq 7 \quad w_3$$

$$-x_1 - x_2 - x_4 + x_5 \leq -7 \quad w_4$$

$$x_1, x_2, x_4, x_5 \geq 0$$

Dual of Revised primal.

$$\text{Minimize } Z^* = 10w_1 - 15w_2 + 7w_4 - 7w_5$$

subject to

$$\begin{aligned} w_1 + w_2 + w_3 - w_4 &\geq 7 \\ -w_1 - w_2 - w_3 + w_4 &\geq -7 \end{aligned}$$

$$w_1, w_2, w_3 \text{ & } w_4 \geq 0$$

Take ~~$w_3 - w_4$~~ $(w_3 - w_4) = w'$ which is unrestricted in sign.

Dual-fund

$$\begin{aligned} \text{Minimize } Z^2 &= 10w_1 - 15w_2 + 7w' \\ \text{subject to } & \end{aligned}$$

$$w_1 - 4w_2 + w' \geq 3$$

$$w_1 - w_2 + w' \geq 4$$

$$w_1 + w_2 + w' = 7$$

$$\begin{aligned} w_1, w_2 &\geq 0 \quad w_1 \text{ & } w_2 \geq 0 \\ \text{& } w' \text{ unrestricted in sign.} \end{aligned}$$

(5)

Dual Simplex method

In simplex method, if one or more solution ($\text{let } z_i \text{ is } x_j$) are negative & optimality condition satisfied, then the solution may be optimum but [as it must satisfy all non-negative constraint] such cases, a variant of the simplex method called dual-simplex method would be used.

Procedure of Dual Simplex method

Step 1: Rewrite the LPP by expressing all the constraints in form and transforms them into equations through slack variables.

Step 2: Express the above problem in the form a tableau. If the optimality condition is satisfied & more basic variables have negative values, the dual simplex method is applicable.

Step 3: Feasibility condition: The basic variable with the most negative value becomes the departing variable (LV). Call the row in which this value is at pivot row. If more than one element for LV choose one.

Step 4: Optimality condition: form ratios by dividing all $C_j - Z_j$ values by the corresponding pivot column.

Step 5: Use elementary row operations to make the pivot element to 1 & then reduce all other elements in the key/pivot column to zero.

Step 6: Repeat steps 3 through 5 until there are no negative values for the basic variables.

(Eq)

Use dual simplex method to solve the following problem

$$\text{Min } Z = 3x_1 + x_2$$

Subject to

$$x_1 + x_2 \geq 1$$

$$2x_1 + 3x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

Sol: Convert the given problem into canonical form

$$Z = Z^*$$

$$\text{Max}(-Z) = -3x_1 - x_2$$

Subject to

$$-x_1 - x_2 \leq -1$$

$$-2x_1 - 3x_2 \leq -2$$

$$x_1, x_2 \geq 0$$

Convert the given problem into Standard form

$$\text{Max}(-Z) = -3x_1 - x_2 + 0s_1 + 0s_2$$

Subject to

$$-x_1 - x_2 + s_1 = -1$$

$$-2x_1 - 3x_2 + s_2 = -2$$

$$x_1, x_2, s_1, s_2 \geq 0$$

Initial solution

Set $x_1 = x_2 = 0 \rightarrow \underline{\text{Non basic variables}}$

Basic variables

$s_1 = -1 \rightarrow$ substituting $x_1 = x_2 = 0$ in constraint

Initial Dual Simplex method

Coefficient of Basic Variable in objective func	C_j					\rightarrow Min Basic variable in objective func $\rightarrow x_1$	x_2	S_1	S_2	Value Basic (C)
		3	-1	0	0					
0	S_1		-1	-1	1	0				-
0	S_2		-2	-3	0	1				-
	Z_j	0	0	0	0	0				
	$C_j - Z_j$	-3	-1	0	0					

$$C_j - Z_j \leq 0 \text{ (optimal)}$$

Solution is optimal but basic variable values are 0 i.e. $b_i < 0$

Solution is optimal but infeasible.

Improve the solution

① outgoing variable: S_2 (as it has most -ve ratio)

Identify in b_i column most negative integer

② Incoming variable: x_2

$$\text{Minimum ratio} = \frac{C_j - Z_j}{-\text{ve value of key row}} = \left(\frac{-3}{-2}, \frac{-1}{-3}, \dots \right) = (1.5, 0.33, \dots)$$

↑
Min

key element = -3.

IInd Dual simplex method

Coefficient of Basic variable in objective func (R _B)	C _j	-3	-1	0	0	R _B val
	Basic Variable (BV)	x ₁	x ₂	s ₁	s ₂	
0	s ₁	-1/3	0	1	-1/3	-1/3
-1	x ₂	2/3	1	0	-1/3	2/3
	Z _j	-2/3	-1	0	1/3	
	C _j - Z _j	-7/3	0	0	-1/3	

$$R_2(\text{new}) = \frac{R_2(\text{old})}{-3}$$

$$\rightarrow \frac{-2}{-3}, \frac{-3}{-3}, \frac{0}{-3}, -\frac{1}{3} \quad \left| \begin{array}{c} b \\ -\frac{2}{-3} \end{array} \right.$$

$$\rightarrow \frac{2}{3}, 1, 0, -\frac{1}{3} \quad \left| \begin{array}{c} 2/3 \end{array} \right.$$

$$R_1(\text{new}) = R_1(\text{old}) + R_2(\text{new})$$

$$R_1(\text{old}) \rightarrow -1 \quad -1 \quad 1 \quad 0 \quad \left| \begin{array}{c} b \\ -1 \end{array} \right.$$

$$R_2(\text{new}) \rightarrow \frac{2}{3} \quad + 1 \quad 0 \quad -\frac{1}{3} \quad \left| \begin{array}{c} 2/3 \end{array} \right.$$

$$R_1(\text{new}) \quad -\frac{1}{3} \quad 0 \quad 1 \quad -\frac{1}{3} \quad -\frac{1}{3} \quad \left| \begin{array}{c} -1 \\ 2/3 \end{array} \right.$$

$C_j - Z_j$ calculations :

$$-3 - (-\frac{2}{3}) = -3 + \frac{2}{3} = \frac{-9+2}{3} = -\frac{7}{3}$$

$$-1 - (-1) = -1 + 1 = 0$$

Improve the solution

3rd Dual Simplex method

Coefficient of B.V in Objective fn (c _B)	G _j Basic var (B.V)	-3 x_1	-1 x_2	0 s_1	0 s_2	Value of Basic var (b _i)
0	s_2	1	0	-3	1	1
-1	x_2	1	1	-1	0	1
	Z_j	-1	-1	1	0	
	$G_j - Z_j$	-2	0	-1	0	

$$R_1(\text{new}) \Rightarrow -3 R_1(\text{old})$$

$$\rightarrow -\frac{1}{3} \times -3 = 1, \quad 0, \quad -3, \quad -\frac{1}{3} \times -3 = 1, \quad -\frac{1}{3}$$

$$B_2(\text{new}) \Rightarrow B_2(\text{old}) + \frac{1}{3} R_1(\text{new})$$

$$B_2(\text{old}) \rightarrow \frac{2}{3} \quad 1 \quad 0 \quad -\frac{1}{3} \quad \left. \right\} \frac{2}{3}$$

$$\begin{aligned} \frac{1}{3} R_1(\text{new}) \rightarrow \frac{1}{3} \times 1 &= \frac{1}{3} \quad \frac{1}{3} \times 0 = 0 \quad \frac{1}{3} \times -3 = -1 \quad \frac{1}{3} \times 1 = \frac{1}{3} \\ \therefore R_1(\text{new}) &= \frac{1}{3} \quad 0 \quad -1 \quad \frac{1}{3} \\ &\hline \end{aligned} \quad \left. \right\} \frac{1}{3} \quad \frac{1}{2}$$

$$B(\text{new}) \quad 1 \quad 1 \quad -1 \quad 0$$

$G_j - Z_j \leq 0$ so the solution is optimal.

& all values of $b_i > 0$

so the solution is optimal & feasible

$$x_1 = 0$$

$$x_2 = 1$$

$$\max(-Z) = -3x_1 - x_2$$

Eg) use dual simplex method to solve the following problem.

Maximize $Z = -2x_1 - 3x_2$

Subject to: $x_1 + x_2 \geq 2$, $2x_1 + x_2 \leq 10$ & $x_1 + x_2 \leq 8$

$x_1, x_2 \geq 0$.

Form:

Canonical form.

$$\text{Max } Z = -2x_1 - 3x_2$$

Subject to

$$-x_1 - x_2 \leq -2$$

$$2x_1 + x_2 \leq 10$$

$$x_1 + x_2 \leq 8$$

$$x_1, x_2 \geq 0$$

std form

$$\text{Max } Z = -2x_1 - 3x_2 + 0s_1 + 0s_2 + 0s_3$$

$$-x_1 - x_2 + s_1 = -2$$

$$2x_1 + x_2 + s_2 = 10$$

$$x_1 + x_2 + s_3 = 8$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

Coefficient of
Basic variable
in objective func
(c_B)

	G_j	Basic Variable (BV)					Value of basic variable (b_i)
		x_1	x_2	s_1	s_2	s_3	
R_1	0	s_1	-1	-1	1	0	0
R_2	0	s_2	2	1	0	1	10
R_3	0	s_3	1	1	0	0	8
	Z_j	0	0	0	0	0	
	$G_j - Z_j$	-2	-3	0	0	0	

Min ratio = $\frac{G_j - Z_j}{\text{-ve element of KR}}$ = $\frac{-2}{-1}, \frac{-3}{-1} = (2, 3)$ left positive min value considered as KC

$$R'_1 = -1 R_1$$

$$R'_2 = R_2 - 2R_1$$

$$R'_3 = R_3 - R_1$$

-2	x_1	1	-1	0	0	2
0	s_2	0	-1	2	1	6
0	s_3	0	0	1	0	6
	Z_j	-2	-2	2	0	-4
	$G_j - Z_j$	0	-1	-2	0	0

Since all $C_j - Z_j$ values are ≤ 0 , & $b_i \geq 0$ the solution is optimal & feasible.

$$\pi_1 = 2, \pi_2 = 0, Z_{\max} = -4.$$

$$\begin{aligned} \text{Max } Z &= -2x_1 - 3x_2 = -2 \times 2 - 3 \times 0 \\ &= -4 - 0 \\ &= \underline{\underline{-4}}. \end{aligned}$$

(eg) use dual simplex method to solve the following problem.

$$\text{Minimize } Z = 2x_1 + \pi_2 + 3x_3$$

$$\text{Subject to } x_1 - 2\pi_2 + x_3 \geq 4$$

$$2x_1 + \pi_2 + x_3 \leq 8$$

$$x_1 - \pi_3 \geq 0$$

$$x_1, \pi_2, x_3 \geq 0.$$

Soln

Canonical form

$$\text{Max } Z = -2x_1 - \pi_2 - 3\pi_2$$

Subject to

$$-x_1 + 2\pi_2 - \pi_3 \leq -4$$

$$2x_1 + \pi_2 + \pi_3 \leq 8$$

$$-x_1 + \pi_3 \leq 0$$

Standard form

$$\text{Max } Z = -2\pi_1 - \pi_2 - 3\pi_2$$

Subject to

$$-\pi_1 + 2\pi_2 - \pi_3 + S_1 = -4$$

$$2\pi_1 + \pi_2 + \pi_3 + S_2 = 8$$

$$-\pi_1 + \pi_3 + S_3 = 0$$

$$\pi_1, \pi_2, \pi_3, S_1, S_2, S_3 \geq 0.$$

C_B	C_j	$B.V$	-2	-1	-3	0	0	0	value of basic var (b_i)
0	S_1		-1	2	-1	1	0	0	-4
0	S_2		2	1	1	0	1	0	8
0	S_3		-1	0	1	0	0	1	0
	π_j		0	0	0	0	0	0	
	$C_j - Z_j$		-2	-1	-3	0	0	0	

$$\text{Min ratio} = \frac{c_j - z_j}{\text{-ve element of KR}} = \left(\frac{-2}{x_1}, \frac{-}{x_2}, \frac{-3}{x_3}, \frac{-}{s_1}, \frac{-}{s_2}, \frac{-}{s_3} \right) \\ = (2, -, 3, -, -, -)$$

↑
RC (least +ve int).

C_B	C_j	$B \cdot V$	x_1	x_2	x_3	s_1	s_2	s_3	Value of basic variable (b_i)
-2	x_1		1	-2	1	-1	0	0	4
0	s_2		0	5	-1	2	1	0	0
0	s_3		0	-2	2	-1	0	1	4
	Z_j		-2	4	-2	2	0	0	-8
	$c_j - z_j$		0	-5	-1	-2	0	0	-

Since $c_j - z_j \geq 0$ & value of $b_i \geq 0$. The solution is optimal and feasible.

$$\therefore x_1 = 4, x_2 = 0, x_3 = 0$$

$$Z_{\max} = -8$$

$$\underline{\underline{Z_{\min} = 8}}$$

(Q) Solve the following LPP using dual simplex method

$$Z_{\min} = 2x_1 + x_2$$

$$\text{Subject to } 3x_1 + x_2 \geq 3$$

$$4x_1 + 3x_2 \geq 6$$

$$x_1 + 2x_2 \geq 3$$

$$\text{And } x_1, x_2 \geq 0.$$

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canonical form

$$Z_{\max} = -2x_1 - x_2$$

subject to

$$-3x_1 - x_2 \leq -3$$

$$-4x_1 - 3x_2 \leq -6$$

$$-x_1 - 2x_2 \leq -3$$

$$\text{Eqs } x_1, x_2 \geq 0$$

std form

$$Z_{\max} = -2x_1 - x_2 + 0s_1 + 0s_2 -$$

$$-3x_1 - x_2 + s_1 = -3$$

$$-4x_1 - 3x_2 + s_2 = -6$$

$$-x_1 - 2x_2 + s_3 = -3$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

C_B	$\frac{C_j}{B^{-1}}$	-2	-1	0	0	0	value of basic vars (b_i)
0	s_1	-3	-1	1	0	0	-3
0	s_2	-4	-3	0	1	0	-6 ←
0	s_3	-1	-2	0	0	1	-3
	Z_j	0	0	0	0	0	
	$G - Z_j$	-2	-1	0	0	0	

$$\text{Min ratio} = \frac{G - Z_j}{\text{re element of ke}} = \left[\frac{-2}{-4}, \frac{-1}{-3}, \dots, \dots \right]$$

x_1	x_2	s_1	s_2	s_3	b_i
0	s_1	$-s_3$	0	1	$-\frac{1}{3}$
-1	x_2	$4s_3$	1	0	$-\frac{1}{3}$
0	s_1	0	0	0	1

$$= \left(\frac{2}{5}, - , - , 1 , - \right)$$

$$= (0.4, - , - , 1 , -)$$

↑ least three integers

C_B	G	-2	-1	0	0	0	b_i
	B.V	x_1	x_2	s_3	s_2	s_3	
-2	x_1	1	0	$-\frac{3}{5}$	$\frac{1}{5}$	0	$\frac{3}{5}$
-1	x_2	0	1	$\frac{4}{5}$	$-\frac{3}{5}$	0	$\frac{6}{5}$
0	s_3	0	0	1	-1	1	0
	Z_j	-2	-1	$\frac{2}{5}$	$\frac{1}{5}$	0	
	$G - Z_j$	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	0	

Since all $G - Z_j \leq 0$ & $b_i \geq 0$ the solution is optimal & feasible.

Hence $x_1 = \frac{3}{5}$, $x_2 = \frac{6}{5}$

$$Z_{\max} = -\frac{12}{5}$$

$$Z_{\min} = \frac{12}{5}$$