MODULE 5

GAME THEORY & GOAL PROGRAMMING

LESSON 1 BASIC CONCEPTS IN GAME THEORY

LESSON OUTLINE

- Introduction to the theory of games
- The definition of a game
- Competitive game
- Managerial applications of the theory of games
- Key concepts in the theory of games
- Types of games

LEARNING OBJECTIVES

After reading this lesson you should be able to

- understand the concept of a game
- grasp the assumptions in the theory of games
- appreciate the managerial applications of the the
- understand the key concepts in the theory of gan
- distinguish between different types of games

Introduction to game theory

Game theory seeks to analyse competing situations which arise out of conflicts of interest. Abraham Maslow's hierarchical model of human needs lays emphasis on fulfilling the basic needs such as food, water, clothes, shelter, air, safety and security. There is conflict of interest between animals and plants in the consumption of natural resources. Animals compete among themselves for securing food. Man competes with animals to earn his food. A man also competes with another man. In the past, nations waged wars to expand the territory of their rule. In the present day world, business organizations compete with each other in getting the market share. The conflicts of interests of human beings are not confined to the basic needs alone. Again considering Abraham Maslow's model of human needs, one can realize that conflicts also arise due to the higher levels of human needs such as love, affection, affiliation, recognition, status, dominance, power, esteem, ego, self-respect, etc. Sometimes one witnesses clashes of ideas of intellectuals also. Every intelligent and rational participant in a conflict wants to be a winner but not all participants can be the winners at a time. The situations of conflict gave birth to Darwin's theory of the 'survival of the fittest'. Nowadays the concepts of conciliation, co-existence, co-operation, coalition and consensus are gaining ground. Game theory is another tool to examine situations of conflict so as to identify the courses of action to be followed and to take appropriate decisions in the long run. Thus this theory assumes importance from managerial perspectives. The pioneering work on the theory of games was done by von Neumann and Morgenstern through their publication entitled 'The Theory of Games and Economic Behaviour' and subsequently the subject was developed by several experts. This theory can offer valuable guidelines to a manager in 'strategic management' which can be used in the decision making process for merger, take-over, joint venture, etc. The results obtained by the application of this theory can serve as an early warning to the top level management in meeting the threats from the competing business organizations and for the conversion of the internal weaknesses and external threats into opportunities and strengths, thereby achieving the goal of maximization of profits. While this theory does not describe any procedure to play a game, it will enable a participant to select the appropriate strategies to be followed in the pursuit of his goals. The situation of failure in a game would activate a participant in the analysis of the relevance of the existing strategies and lead him to identify better, novel strategies for the future occasions.

Definitions of game theory

There are several definitions of game theory. A few standard definitions are presented below.

In the perception of Robert Mockler, "Game theory is a mathematical technique helpful in making decisions in situations of conflicts, where the success of one part depends at the expense of others, and where the individual decision maker is not in complete control of the factors influencing the outcome".

The definition given by William G. Nelson runs as follows: "Game theory, more properly **the theory of games of strategy**, is a mathematical method of analyzing a conflict. The alternative is not between this decision or that decision, but between this strategy or that strategy to be used against the conflicting interest".

In the opinion of Matrin Shubik, "Game theory is a method of the study of decision making in situation of conflict. It deals with human processes in which the individual decision-unit is not in complete control of other decision-units entering into the environment".

According to von Neumann and Morgenstern, "The 'Game' is simply the totality of the rules which describe it. Every particular instance at which the game is played – in a particular way – from beginning to end is a 'play'. The game consists of a sequence of moves, and the play of a sequence of choices".

J.C.C McKinsey points out a valid distinction between two words, namely 'game' and 'play'. According to him, "game refers to a particular realization of the rules".

In the words of O.T. Bartos, "The theory of games can be used for 'prescribing' how an intelligent person should go about resolving social conflicts, ranging all the way from open warfare between nations to disagreements between husband and wife".

Martin K Starr gave the following definition: "Management models in the competitive sphere are usually termed game models. By studying game theory, we can obtain substantial information into management's role under competitive conditions, even though much of the game theory is neither directly operational nor implementable".

According to Edwin Mansfield, "A game is a competitive situation where two or more persons pursue their own interests and no person can dictate the outcome. Each player, an entity with the same interests, make his own decisions. A player can be an individual or a group".

Assumptions for a Competitive Game

Game theory helps in finding out the best course of action for a firm in view of the anticipated countermoves from the competing organizations. A competitive situation is a competitive game if the following properties hold:

1. The number of competitors is finite, say N.
2. A finite set of possible courses of action is available to each of the N competitors.
3. A play of the game results when each competitor selects a course of action from the set of courses available to him. In game theory we make an important assumption that al the players select their courses of action simultaneously. As a result, no competitor will be in a position to know the choices of his competitors.
4. The outcome of a play consists of the particular courses of action chosen by the individual players. Each outcome leads to a set of payments, one to each player, which may be either positive, or negative, or zero.

Managerial Applications of the Theory of Games

The techniques of game theory can be effectively applied to various managerial problems as detailed below:

1) Analysis of the market strategies of a business organization in the long run.
2) Evaluation of the responses of the consumers to a new product.
3) Resolving the conflict between two groups in a business organization.
4) Decision making on the techniques to increase market share.

5) Material procurement process.
6) Decision making for transportation problem.
7) Evaluation of the distribution system.
8) Evaluation of the location of the facilities.
9) Examination of new business ventures and
10) Competitive economic environment.

Key concepts in the Theory of Games

Several of the key concepts used in the theory of games are described below:

Players:

The competitors or decision makers in a game are called the players of the game.

Strategies:

The alternative courses of action available to a player are referred to as his strategies.

Pay off:

The outcome of playing a game is called the pay off to the concerned player.

Optimal Strategy:

A strategy by which a player can achieve the best pay off is called the optimal strategy for him.

Zero-sum game:

A game in which the total payoffs to all the players at the end of the game is zero is referred to as a zero-sum game.

Non-zero sum game:

Games with "less than complete conflict of interest" are called non-zero sum games. The problems faced by a large number of business organizations come under this category. In such games, the gain of one player in terms of his success need not be completely at the expense of the other player.

Payoff matrix:

The tabular display of the payoffs to players under various alternatives is called the payoff matrix of the game.

Pure strategy:

If the game is such that each player can identify one and only one strategy as the optimal strategy in each play of the game, then that strategy is referred to as the best strategy for that player and the game is referred to as a game of pure strategy or a pure game.

Mixed strategy:

If there is no one specific strategy as the ‘best strategy’ for any player in a game, then the game is referred to as a game of mixed strategy or a mixed game. In such a game, each player has to choose different alternative courses of action from time to time.

N-person game:

A game in which N-players take part is called an N-person game.

Maximin-Minimax Principle :

The maximum of the minimum gains is called the maximin value of the game and the corresponding strategy is called the maximin strategy. Similarly the minimum of the maximum losses is called the minimax value of the game and the corresponding strategy is called the minimax strategy. If both the values are equal, then that would guarantee the best of the worst results.

Negotiable or cooperative game:

If the game is such that the players are taken to cooperate on any or every action which may increase the payoff of either player, then we call it a negotiable or cooperative game.

Non-negotiable or non-cooperative game:

If the players are not permitted for coalition then we refer to the game as a non-negotiable or non-cooperative game.

Saddle point:

A saddle point of a game is that place in the payoff matrix where the maximum of the row minima is equal to the minimum of the column maxima. The payoff at the saddle point is called **the value of the game** and the corresponding strategies are called the **pure strategies.**

Dominance:

One of the strategies of either player may be inferior to at least one of the remaining ones. The superior strategies are said to dominate the inferior ones.

Types of Games:

There are several classifications of a game. The classification may be based on various factors such as the number of participants, the gain or loss to each participant, the number of strategies available to each participant, etc. Some of the important types of games are enumerated below.

Two person games and n-person games:

In two person games, there are exactly two players and each competitor will have a finite number of strategies. If the number of players in a game exceeds two, then we refer to the game as n-person game.

Zero sum game and non-zero sum game:

If the sum of the payments to all the players in a game is zero for every possible outcome of the game, then we refer to the game as a zero sum game. If the sum of the payoffs from any play of the game is either positive or negative but not zero, then the game is called a non-zero sum game

Games of perfect information and games of imperfect information:

A game of perfect information is the one in which each player can find out the strategy that would be followed by his opponent. On the other hand, a game of imperfect information is the one in which no player can know in advance what strategy would be adopted by the competitor and a player has to proceed in his game with his guess works only.

Games with finite number of moves / players and games with unlimited number of moves:

A game with a finite number of moves is the one in which the number of moves for each player is limited before the start of the play. On the other hand, if the game can be continued over an extended period of time and the number of moves for any player has no restriction, then we call it a game with unlimited number of moves.

Constant-sum games:

If the sum of the game is not zero but the sum of the payoffs to both players in each case is constant, then we call it a constant sum game. It is possible to reduce such a game to a zero-sum game.

2x2 two person game and 2xn and mx2 games:

When the number of players in a game is two and each player has exactly two strategies, the game is referred to as 2x2 two person game.

A game in which the first player has precisely two strategies and the second player has three or more strategies is called an 2xn game.

A game in which the first player has three or more strategies and the second player has exactly two strategies is called an mx2 game.

3x3 and large games:

When the number of players in a game is two and each player has exactly three strategies, we call it a 3x3 two person game.

Two-person zero sum games are said to be larger if each of the two players has 3 or more choices.

The examination of 3x3 and larger games is involves difficulties. For such games, the technique of linear programming can be used as a method of solution to identify the optimum strategies for the two players.

Non-constant games :

Consider a game with two players. If the sum of the payoffs to the two players is not constant in all the plays of the game, then we call it a non-constant game.

Such games are divided into negotiable or cooperative games and non-negotiable or non-cooperative games.

QUESTIONS

1. Explain the concept of a game.
2. Define a game.
3. State the assumptions for a competitive game.
4. State the managerial applications of the theory of games.
5. Explain the following terms: strategy, pay-off matrix, saddle point, pure strategy and mixed strategy.
6. Explain the following terms: two person game, two person zero sum game, value of a game, 2xn game and mx2 game.

LESSON 2

TWO-PERSON ZERO SUM GAMES

LESSON OUTLINE

- The concept of a two-person zero sum game
- The assumptions for a two-person zero sum game
- Minimax and Maximin principles

LEARNING OBJECTIVES

After reading this lesson you should be able to

- understand the concept of a two-person zero sum game
- have an idea of the assumptions for a two-person zero sum game
- understand Minimax and Maximin principles
- solve a two-person zero sum game
- interpret the results from the payoff matrix of a two-person zero sum game

Definition of two-person zero sum game

A game with only two players, say player A and player B, is called a two-person zero sum game if the gain of the player A is equal to the loss of the player B, so that the total sum is zero.

Payoff matrix:

When players select their particular strategies, the payoffs (gains or losses) can be represented in the form of a payoff matrix.

Since the game is zero sum, the gain of one player is equal to the loss of other and vice-versa. Suppose A has m strategies and B has n strategies. Consider the following payoff matrix.

Player B's strategies

$$\text{Player A's strategies} \quad \begin{array}{c} \\ A_1 \\ A_1 \\ \vdots \\ Am \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

Player A wishes to gain as large a payoff a_{ij} as possible while player B will do his best to reach as small a value a_{ij} as possible where the gain to player B and loss to player A be (-a_{ij}).

Assumptions for two-person zero sum game:

For building any model, certain reasonable assumptions are quite necessary. Some assumptions for building a model of two-person zero sum game are listed below.

a) Each player has available to him a finite number of possible courses of action. Sometimes the set of courses of action may be the same for each player. Or, certain courses of action may be available to both players while each player may have certain specific courses of action which are not available to the other player.

b) Player A attempts to maximize gains to himself. Player B tries to minimize losses to himself.

c) The decisions of both players are made individually prior to the play with no communication between them.

d) The decisions are made and announced simultaneously so that neither player has an advantage resulting from direct knowledge of the other player's decision.

e) Both players know the possible payoffs of themselves and their opponents.

Minimax and Maximin Principles

The selection of an optimal strategy by each player without the knowledge of the competitor's strategy is the basic problem of playing games.

The objective of game theory is to know how these players must select their respective strategies, so that they may optimize their payoffs. Such a criterion of decision making is referred to as minimax-maximin principle. This principle in games of pure strategies leads to the best possible selection of a strategy for both players.

For example, if player A chooses his i^{th} strategy, then he gains at least the payoff min a_{ij}, which is minimum of the i^{th} row elements in the payoff matrix. Since his objective is to maximize his payoff, he can choose strategy *i* so as to make his payoff as large as possible. i.e., a payoff which is not less than $\max\limits_{1\le i\le m}\ \min\limits_{1\le j\le n}\ a_{ij}$.

Similarly player B can choose j^{th} column elements so as to make his loss not greater than $\min\limits_{1\le j\le n}\ \max\limits_{1\le i\le m} a_{ij}$.

If the maximin value for a player is equal to the minimax value for another player, i.e.

$$\max_{1\le i\le m}\ \min_{1\le j\le n}\ a_{ij} = V = \min_{1\le j\le n}\ \max_{1\le i\le m}\ a_{ij}$$

then the game is said to have a saddle point (equilibrium point) and the corresponding strategies are called optimal strategies. If there are two or more saddle points, they must be equal.

The amount of payoff, i.e., V at an equilibrium point is known as the **value of the game**.

The optimal strategies can be identified by the players in the long run.

Fair game:

The game is said to be fair if the value of the game $V = 0$.

Problem 1:

Solve the game with the following pay-off matrix.

Player B

Strategies

$$
\text{Player A Strategies}\quad
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ 4
\end{array}
\begin{array}{c}
\begin{array}{ccccc} I & II & III & IV & V \end{array} \\
\left|\begin{array}{ccccc}
-2 & 5 & -3 & 6 & 7 \\
4 & 6 & 8 & -1 & 6 \\
8 & 2 & 3 & 5 & 4 \\
15 & 14 & 18 & 12 & 20
\end{array}\right|
\end{array}
$$

Solution:

First consider the minimum of each row.

Row	Minimum Value
1	-3
2	-1
3	2
4	12

Maximum of {-3, -1, 2, 12} = 12

Next consider the maximum of each column.

Column	Maximum Value
1	15
2	14
3	18
4	12
5	20

Minimum of {15, 14, 18, 12, 20}= 12

We see that the maximum of row minima = the minimum of the column maxima. So the game has a saddle point. The common value is 12. Therefore the value V of the game = 12.

Interpretation:

In the long run, the following best strategies will be identified by the two players:

The best strategy for player A is strategy 4.

The best strategy for player B is strategy IV.

The game is favourable to player A.

Problem 2:

Solve the game with the following pay-off matrix

Player Y

Strategies

$$
\begin{array}{cc|ccccc|}
 & & I & II & III & IV & V \\
\hline
 & 1 & 9 & 12 & 7 & 14 & 26 \\
\text{Player X Strategies} & 2 & 25 & 35 & 20 & 28 & 30 \\
 & 3 & 7 & 6 & -8 & 3 & 2 \\
 & 4 & 8 & 11 & 13 & -2 & 1 \\
\hline
\end{array}
$$

Solution:

First consider the minimum of each row.

Row	Minimum Value
1	7
2	20
3	-8
4	-2

Maximum of {7, 20, –8, -2} = 20

Next consider the maximum of each column.

Column	Maximum Value
1	25
2	35
3	20
4	28
5	30

Minimum of {25, 35, 20, 28, 30}= 20

It is observed that the maximum of row minima and the minimum of the column maxima are equal. Hence the given the game has a saddle point. The common value is 20. This indicates that the value V of the game is 20.

Interpretation.

The best strategy for player X is strategy 2.

The best strategy for player Y is strategy III.

The game is favourable to player A.

Problem 3:

Solve the following game:

Player B

Strategies

$$\text{Player A Strategies} \quad \begin{array}{c|cccc|} & I & II & III & IV \\ \hline 1 & 1 & -6 & 8 & 4 \\ 2 & 3 & -7 & 2 & -8 \\ 3 & 5 & -5 & -1 & 0 \\ 4 & 3 & -4 & 5 & 7 \\ \hline \end{array}$$

Solution

First consider the minimum of each row.

Row	Minimum Value
1	-6
2	-8
3	-5
4	-4

Maximum of {-6, -8, -5, -4} = -4

Next consider the maximum of each column.

Column	Maximum Value
1	5
2	-4
3	8
4	7

Minimum of {5, -4, 8, 7}= - 4

Since the max {row minima} = min {column maxima}, the game under consideration has a saddle point. The common value is –4. Hence the value of the game is –4.

Interpretation.

The best strategy for player A is strategy 4.

The best strategy for player B is strategy II.

Since the value of the game is negative, it is concluded that the game is favourable to player B.

QUESTIONS

1. What is meant by a two-person zero sum game? Explain.
2. State the assumptions for a two-person zero sum game.
3. Explain Minimax and Maximin principles.
4. How will you interpret the results from the payoff matrix of a two-person zero sum game? Explain.
5. What is a fair game? Explain.
6. Solve the game with the following pay-off matrix.

Player B

Strategies

$$\begin{array}{ll} & \begin{array}{ccccc} I & II & III & IV & V \end{array} \\ \text{Player A Strategies} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \left|\begin{array}{ccccc} 7 & 5 & 2 & 3 & 9 \\ 10 & 8 & 7 & 4 & 5 \\ 9 & 12 & 0 & 2 & 1 \\ 11 & -2 & -1 & 3 & 4 \end{array}\right| \end{array}$$

Answer: Best strategy for A: 2

Best strategy for B: IV

V = 4

The game is favourable to player A

7. Solve the game with the following pay-off matrix.

Player B

Strategies

$$\begin{array}{ll} & \begin{array}{ccccc} I & II & III & IV & V \end{array} \\ \text{Player A Strategies} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \left|\begin{array}{ccccc} -2 & -3 & 8 & 7 & 0 \\ 1 & -7 & -5 & -2 & 3 \\ 4 & -2 & 3 & 5 & -1 \\ 6 & -4 & 5 & 4 & 7 \end{array}\right| \end{array}$$

Answer: Best strategy for A: 3

Best strategy for B: II

V = -2

The game is favourable to player B

8. Solve the game with the following pay-off matrix.

Player B

$$\text{Player A} \quad \begin{bmatrix} 6 & 4 \\ 7 & 5 \end{bmatrix}$$

Answer: Best strategy for A: 2

Best strategy for B: II

V = 5

The game is favourable to player A

9. Solve the following game and interpret the result.

Player B

Strategies

$$\begin{array}{cc} & \begin{array}{cccc} I & II & III & IV \end{array} \\ \text{Player A Strategies} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} -3 & -7 & 1 & 3 \\ -1 & 2 & -3 & 1 \\ 0 & 4 & 2 & 6 \\ -2 & -1 & -5 & 1 \end{bmatrix} \end{array}$$

Answer: Best strategy for A: 3

Best strategy for B: I

V = 0

The value V = 0 indicates that the game is a fair one.

10. Solve the following game:

Player B

Strategies

$$\begin{array}{cc} & \begin{array}{ccc} I & II & III \end{array} \\ \text{Player A Strategies} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 8 & 2 \\ 3 & 5 & 6 \\ 2 & 2 & 1 \end{bmatrix} \end{array}$$

Answer: Best strategy for A: 2

Best strategy for B: I

V = 3

The game is favourable to player A

11. Solve the game

Player B

$$
\text{Player A}\quad
\begin{array}{c}
 \\ 1 \\ 2 \\ 3
\end{array}
\begin{array}{c}
\begin{array}{cccc} \mathit{I} & \mathit{II} & \mathit{III} & \mathit{IV} \end{array} \\
\left|\begin{array}{cccc}
4 & -1 & 2 & 0 \\
-3 & -5 & -9 & -2 \\
2 & -8 & 0 & -11
\end{array}\right|
\end{array}
$$

Answer : V = -1

12. Solve the game

Player Y

$$
\text{Player X}\quad
\begin{array}{c|ccccc|}
 & \mathit{I} & \mathit{II} & \mathit{III} & \mathit{IV} & \mathit{V} \\
\hline
1 & 4 & 0 & 1 & 7 & -1 \\
2 & 0 & -3 & -5 & -7 & 5 \\
3 & 3 & 2 & 3 & 4 & 3 \\
4 & -6 & 4 & -1 & 0 & 5 \\
5 & 0 & 0 & 6 & 0 & 0 \\
\hline
\end{array}
$$

Answer **:** V = 2

13. Solve the game

Player B

$$
\text{Player A}\quad
\begin{array}{c|ccccc|}
 & \mathit{I} & \mathit{II} & \mathit{III} & \mathit{IV} & \mathit{V} \\
\hline
1 & 9 & 3 & 4 & 4 & 2 \\
2 & 8 & 6 & 8 & 5 & 12 \\
3 & 10 & 7 & 19 & 18 & 14 \\
4 & 8 & 6 & 8 & 11 & 6 \\
5 & 3 & 5 & 16 & 10 & 8 \\
\hline
\end{array}
$$

Answer **:** V = 7

14. Solve the game

Player Y

Player X

17	10	12	5	4	8
2	5	6	7	6	9
7	6	9	2	3	1
10	11	14	8	13	8
20	18	17	10	15	17
12	11	15	9	5	11

Answer **:** V = 10

15. Solve the game

Player B

Player A

12	14	8	7	4	9
2	13	6	7	9	8
13	6	8	6	3	1
14	9	10	8	9	6
20	18	17	11	14	16
8	12	16	9	6	13

Answer : V = 11

16. Examine whether the following game is fair.

Player Y

Player X

6	–4	–3	–2
3	5	0	8
7	–2	–6	5

Answer : V = 0. Therefore, it is a fair game.

LESSON 3

GAMES WITH NO SADDLE POINT

LESSON OUTLINE

- The concept of a 2x2 game with no saddle point
- The method of solution

LEARNING OBJECTIVES

After reading this lesson you should be able to

- understand the concept of a 2x2 game with no saddle point
- know the method of solution of a 2x2 game without saddle point
- solve a game with a given payoff matrix
- interpret the results obtained from the payoff matrix

2 x 2 zero-sum game

When each one of the first player A and the second player B has exactly two strategies, we have a 2 x 2 game.

Motivating point

First let us consider an illustrative example.

Problem 1:

Examine whether the following 2 x 2 game has a saddle point

$$\begin{array}{cc} & \textbf{Player B} \\ \textbf{Player A} & \begin{vmatrix} 3 & 5 \\ 4 & 2 \end{vmatrix} \end{array}$$

Solution:

First consider the minimum of each row.

Row	Minimum Value
1	3
2	2

Maximum of {3, 2} = 3

Next consider the maximum of each column.

Column	Maximum Value
1	4
2	5

Minimum of {4, 5}= 4

We see that max {row minima} and min {column maxima} are not equal. Hence the game has no saddle point.

Method of solution of a 2x2 zero-sum game without saddle point

Suppose that a 2x2 game has no saddle point. Suppose the game has the following pay-off matrix.

Player B

Strategy

$$\textbf{Player A Strategy} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Since this game has no saddle point, the following condition shall hold:

$$Max\{Min\{a,b\}, Min\{c,d\}\} \neq Min\{Max\{a,c\}, Max\{b,d\}\}$$

In this case, the game is called a mixed game**. No strategy of Player A can be called the best strategy for him. Therefore A has to use both of his strategies. Similarly no strategy of Player B can be called the best strategy for him and he has to use both of his strategies.**

Let *p* be the probability that Player A will use his first strategy. Then the probability that Player A will use his second strategy is 1-p.

<u>If Player B follows his first strategy</u>

$$\text{Expected value of the pay-off to Player A}$$
$$=\left\{\begin{matrix}\text{Expected value of the pay-off to Player A}\\ \text{arising from his first strategy}\end{matrix}\right\}+\left\{\begin{matrix}\text{Expected value of the pay-off to Player A}\\ \text{arising from his second strategy}\end{matrix}\right\}$$

$$=ap+c(1-p) \qquad (1)$$

In the above equation, note that the expected value is got as the product of the corresponding values of the pay-off and the probability.

<u>If Player B follows his second strategy</u>

$$\left.\begin{matrix}\text{Expected value of the}\\ \text{pay-off to Player A}\end{matrix}\right\}=bp+d(1-p) \qquad (2)$$

If the expected values in equations (1) and (2) are different, Player B will prefer the minimum of the two expected values that he has to give to player A. Thus B will have a pure strategy. This contradicts our assumption that the game is a mixed one. Therefore the expected values of the pay-offs to Player A in equations (1) and (2) should be equal. Thus we have the condition

$$
\begin{aligned}
ap + c(1-p) &= bp + d(1-p) \\
ap - bp &= (1-p)[d-c] \\
p(a-b) &= (d-c) - p(d-c) \\
p(a-b) + p(d-c) &= d-c \\
p(a-b+d-c) &= d-c \\
p &= \frac{d-c}{(a+d)-(b+c)} \\
1-p &= \frac{a+d-b-c-d+c}{(a+d)-(b+c)} \\
&= \frac{a-b}{(a+d)-(b+c)}
\end{aligned}
$$

$$
\left\{\begin{matrix}\text{The number of times A} \\ \text{will use first strategy}\end{matrix}\right\} : \left\{\begin{matrix}\text{The number of times A} \\ \text{will use second strategy}\end{matrix}\right\} = \frac{d-c}{(a+d)-(b+c)} : \frac{a-b}{(a+d)-(b+c)}
$$

The expected pay-off to Player A

$$
\begin{aligned}
&= ap + c(1-p) \\
&= c + p(a-c) \\
&= c + \frac{(d-c)(a-c)}{(a+d)-(b+c)} \\
&= \frac{c\{(a+d)-(b+c)\} + (d-c)(a-c)}{(a+d)-(b+c)} \\
&= \frac{ac + cd - bc - c^2 + ad - cd - ac + c^2)}{(a+d)-(b+c)} \\
&= \frac{ad - bc}{(a+d)-(b+c)}
\end{aligned}
$$

Therefore, the value V of the game is

$$\frac{ad-bc}{(a+d)-(b+c)}$$

To find the number of times that B will use his first strategy and second strategy:

Let the probability that B will use his first strategy be *r*. Then the probability that B will use his second strategy is *1-r*.

<u>When A use his first strategy</u>

The expected value of loss to Player B with his first strategy = ar

The expected value of loss to Player B with his second strategy = $b(1-r)$

Therefore the expected value of loss to B = $ar + b(1-r)$ (3)

<u>When A use his second strategy</u>

The expected value of loss to Player B with his first strategy = cr

The expected value of loss to Player B with his second strategy = *d(1-r)*

Therefore the expected value of loss to B = c*r* + *d(1-r)* (4)

If the two expected values are different then it results in a pure game, which is a contradiction. Therefore the expected values of loss to Player B in equations (3) and (4) should be equal. Hence we have the condition

$$
\begin{aligned}
ar + b(1-r) &= cr + d(1-r) \\
ar + b - br &= cr + d - dr \\
ar - br - cr + dr &= d - b \\
r(a-b-c+d) &= d - b \\
r &= \frac{d-b}{a-b-c+d} \\
&= \frac{d-b}{(a+d)-(b+c)}
\end{aligned}
$$

Problem 2:

Solve the following game

$$
\begin{array}{cc} & Y \\ X & \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix} \end{array}
$$

Solution:

First consider the row minima.

Row	Minimum Value
1	2
2	1

Maximum of {2, 1} = 2

Next consider the maximum of each column.

Column	Maximum Value
1	4
2	5

Minimum of {4, 5}= 4

We see that

Max {row minima} $\neq$ min {column maxima}

So the game has no saddle point. Therefore it is a mixed game.

We have a = 2, b = 5, c = 4 and d = 1.

Let *p* be the probability that player X will use his first strategy. We have

$$p = \frac{d-c}{(a+d)-(b+c)}$$
$$= \frac{1-4}{(2+1)-(5+4)}$$
$$= \frac{-3}{3-9}$$
$$= \frac{-3}{-6}$$
$$= \frac{1}{2}$$

The probability that player *X* will use his second strategy is $1-p = 1-\frac{1}{2} = \frac{1}{2}$.

Value of the game V = $\frac{ad-bc}{(a+d)-(b+c)} = \frac{2-20}{3-9} = \frac{-18}{-6} = 3$.

Let *r* be the probability that Player Y will use his first strategy. Then the probability that Y will use his second strategy is *(1-r)*. We have

$$r = \frac{d-b}{(a+d)-(b+c)}$$
$$= \frac{1-5}{(2+1)-(5+4)}$$
$$= \frac{-4}{3-9}$$
$$= \frac{-4}{-6}$$
$$= \frac{2}{3}$$
$$1-r = 1-\frac{2}{3} = \frac{1}{3}$$

Interpretation.

p : (1-p) = $\frac{1}{2} : \frac{1}{2}$

Therefore, out of 2 trials, player X will use his first strategy once and his second strategy once.

$r : (1\text{-}r) = \frac{2}{3} : \frac{1}{3}$

Therefore, out of 3 trials, player Y will use his first strategy twice and his second strategy once.

QUESTIONS

1. What is a 2x2 game with no saddle point? Explain.
2. Explain the method of solution of a 2x2 game without saddle point.
3. Solve the following game

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} 12 & 4 \\ 3 & 7 \end{bmatrix} \end{array}$$

   Answer: $p = \frac{1}{3}$, $r = \frac{1}{4}$, $V = 6$

4. Solve the following game

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} 5 & -4 \\ -9 & 3 \end{bmatrix} \end{array}$$

   Answer: $p = \frac{4}{7}$, $r = \frac{1}{3}$, $V = -1$

5. Solve the following game

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} 10 & 4 \\ 6 & 8 \end{bmatrix} \end{array}$$

   Answer: $p = \frac{1}{4}$, $r = \frac{1}{2}$, $V = 7$

6. Solve the following game

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} 20 & 8 \\ -2 & 10 \end{bmatrix} \end{array}$$

   Answer: $p = \frac{1}{2}$, $r = \frac{1}{12}$, $V = 9$

7. Solve the following game

Y

$$\begin{array}{cc} X & \begin{bmatrix} 10 & 2 \\ 1 & 5 \end{bmatrix} \end{array}$$

Answer: $p = \frac{1}{3}$, $r = \frac{1}{4}$, $V = 4$

8. Solve the following game

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} 12 & 6 \\ 6 & 9 \end{bmatrix} \end{array}$$

Answer: $p = \frac{1}{3}$, $r = \frac{1}{3}$, $V = 8$

9. Solve the following game

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} 10 & 8 \\ 8 & 10 \end{bmatrix} \end{array}$$

Answer: $p = \frac{1}{2}$, $r = \frac{1}{2}$, $V = 9$

10. Solve the following game

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} 16 & 4 \\ 4 & 8 \end{bmatrix} \end{array}$$

Answer: $p = \frac{1}{4}$, $r = \frac{1}{4}$, $V = 7$

11. Solve the following game

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} -11 & 5 \\ 7 & -9 \end{bmatrix} \end{array}$$

Answer: $p = \frac{1}{2}$, $r = \frac{7}{16}$, $V = -2$

12. Solve the following game

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} -9 & 3 \\ 5 & -7 \end{bmatrix} \end{array}$$

Answer: $p = \frac{1}{2}$, $r = \frac{5}{12}$, $V = -2$

LESSON 4
THE PRINCIPLE OF DOMINANCE

LESSON OUTLINE

- The principle of dominance
- Dividing a game into sub games

LEARNING OBJECTIVES

After reading this lesson you should be able to

- understand the principle of dominance
- solve a game using the principle of dominance
- solve a game by dividing a game into sub games

The principle of dominance

In the previous lesson, we have discussed the method of solution of a game without a saddle point. While solving a game without a saddle point, one comes across the phenomenon of the dominance of a row over another row or a column over another column in the pay-off matrix of the game. Such a situation is discussed in the sequel.

In a given pay-off matrix A, we say that the i^{th} row dominates the k^{th} row if

$$a_{ij} \geq a_{kj} \quad \text{for all } j = 1,2,\ldots,n$$

and

$$a_{ij} > a_{kj} \quad \text{for at least one } j.$$

In such a situation player A will never use the strategy corresponding to k^{th} row, because he will gain less for choosing such a strategy.

Similarly, we say the p^{th} column in the matrix dominates the q^{th} column if

$$a_{ip} \leq a_{iq} \quad \text{for all } i = 1,2,\ldots,m$$

and

$$a_{ip} < a_{iq} \quad \text{for at least one } i.$$

In this case, the player B will loose more by choosing the strategy for the q^{th} column than by choosing the strategy for the p^{th} column. So he will never use the strategy corresponding to the q^{th} column. When dominance of a row (or a column) in the pay-off matrix occurs, we can delete a row (or a column) from that matrix and arrive at a reduced matrix. This principle of dominance can be used in the determination of the solution for a given game.

Let us consider an illustrative example involving the phenomenon of dominance in a game.

Problem 1:

Solve the game with the following pay-off matrix:

$$\text{Player A}\quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \text{Player B} \\ \begin{array}{cccc} I & II & III & IV \end{array} \\ \begin{bmatrix} 4 & 2 & 3 & 6 \\ 3 & 4 & 7 & 5 \\ 6 & 3 & 5 & 4 \end{bmatrix} \end{array}$$

Solution:

First consider the minimum of each row.

Row	Minimum Value
1	2
2	3
3	3

Maximum of {2, 3, 3} = 3

Next consider the maximum of each column.

Column	Maximum Value
1	6
2	4
3	7
4	6

Minimum of {6, 4, 7, 6}= 4

The following condition holds:

$$\text{Max \{row minima\}} \neq \text{min \{column maxima\}}$$

Therefore we see that there is no saddle point for the game under consideration.

Compare columns II and III.

Column II	Column III
2	3
4	7
3	5

We see that each element in column III is greater than the corresponding element in column II. The choice is for player B. Since column II dominates column III, player B will discard his strategy 3.

Now we have the reduced game

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} I & II & IV \end{array} \\ \begin{bmatrix} 4 & 2 & 6 \\ 3 & 4 & 5 \\ 6 & 3 & 4 \end{bmatrix} \end{array}$$

For this matrix again, there is no saddle point. Column II dominates column IV. The choice is for player B. So player B will give up his strategy 4

The game reduces to the following:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} I & II \end{array} \\ \begin{bmatrix} 4 & 2 \\ 3 & 4 \\ 6 & 3 \end{bmatrix} \end{array}$$

This matrix has no saddle point.

The third row dominates the first row. The choice is for player A. He will give up his strategy 1 and retain strategy 3. The game reduces to the following:

$$\begin{bmatrix} 3 & 4 \\ 6 & 3 \end{bmatrix}$$

Again, there is no saddle point. We have a 2x2 matrix. Take this matrix as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then we have a = 3, b = 4, c = 6 and d = 3. Use the formulae for p, 1-p, r, 1-r and V.

$$\begin{aligned} p &= \frac{d-c}{(a+d)-(b+c)} \\ &= \frac{3-6}{(3+3)-(6+4)} \\ &= \frac{-3}{6-10} \\ &= \frac{-3}{-4} \\ &= \frac{3}{4} \end{aligned}$$

$$1-p = 1-\frac{3}{4} = \frac{1}{4}$$

$$r = \frac{d-b}{(a+d)-(b+c)}$$

$$= \frac{3-4}{(3+3)-(6+4)}$$

$$= \frac{-1}{6-10}$$

$$= \frac{-1}{-4}$$

$$= \frac{1}{4}$$

$$1-r = 1-\frac{1}{4} = \frac{3}{4}$$

The value of the game

$$V = \frac{ad-bc}{(a+d)-(b+c)}$$

$$= \frac{3x3-4x6}{-4}$$

$$= \frac{-15}{-4}$$

$$= \frac{15}{4}$$

Thus, X = $\left(\frac{3}{4}, \frac{1}{4}, 0, 0\right)$ and Y = $\left(\frac{1}{4}, \frac{3}{4}, 0, 0\right)$ are the optimal strategies.

Method of convex linear combination

A strategy, say s, can also be dominated if it is inferior to a convex linear combination of several other pure strategies. In this case if the domination is strict, then the strategy s can be deleted. If strategy s dominates the convex linear combination of some other pure strategies, then one of the pure strategies involved in the combination may be deleted. The domination will be decided as per the above rules. Let us consider an example to illustrate this case.

Problem 2:

Solve the game with the following pay-off matrix for firm A:

Firm B

$$\text{Firm A} \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\ \begin{bmatrix} 4 & 8 & -2 & 5 & 6 \\ 4 & 0 & 6 & 8 & 5 \\ -2 & -6 & -4 & 4 & 2 \\ 4 & -3 & 5 & 6 & 3 \\ 4 & -1 & 5 & 7 & 3 \end{bmatrix} \end{array}$$

Solution:

First consider the minimum of each row.

Row	Minimum Value
1	-2
2	0
3	-6
4	-3
5	-1

Maximum of {-2, 0, -6, -3, -1} = 0

Next consider the maximum of each column.

Column	Maximum Value
1	4
2	8
3	6
4	8
5	6

Minimum of { 4, 8, 6, 8, 6}= 4

Hence,

Maximum of {row minima} $\neq$ minimum of {column maxima}.

So we see that there is no saddle point. Compare the second row with the fifth row. Each element in the second row exceeds the corresponding element in the fifth row. Therefore, A_2 dominates A_5. The choice is for firm A. It will retain strategy A_2 and give up strategy A_5. Therefore the game reduces to the following.

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\ \begin{bmatrix} 4 & 8 & -2 & 5 & 6 \\ 4 & 0 & 6 & 8 & 5 \\ -2 & -6 & -4 & 4 & 2 \\ 4 & -3 & 5 & 6 & 3 \end{bmatrix} \end{array}$$

Compare the second and fourth rows. We see that A_2 dominates A_4. So, firm A will retain the strategy A_2 and give up the strategy A_4. Thus the game reduces to the following:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\ \begin{bmatrix} 4 & 8 & -2 & 5 & 6 \\ 4 & 0 & 6 & 8 & 5 \\ -2 & -6 & -4 & 4 & 2 \end{bmatrix} \end{array}$$

Compare the first and fifth columns. It is observed that B_1 dominates B_5. The choice is for firm B. It will retain the strategy B_1 and give up the strategy B_5. Thus the game reduces to the following

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 4 & 8 & -2 & 5 \\ 4 & 0 & 6 & 8 \\ -2 & -6 & -4 & 4 \end{bmatrix} \end{array}$$

Compare the first and fourth columns. We notice that B_1 dominates B_4. So firm B will discard the strategy B_4 and retain the strategy B_1. Thus the game reduces to the following:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 4 & 8 & -2 \\ 4 & 0 & 6 \\ -2 & -6 & -4 \end{bmatrix} \end{array}$$

For this reduced game, we check that there is no saddle point.

Now none of the pure strategies of firms A and B is inferior to any of their other strategies. But, we observe that convex linear combination of the strategies B_2 and B_3 dominates B_1, i.e. the averages of payoffs due to strategies B_2 and B_3,

$$\left\{\frac{8-2}{2}, \frac{0+6}{2}, \frac{-6-4}{2}\right\} = \{3, 3, -5\}$$

dominate B_1. Thus B_1 may be omitted from consideration. So we have the reduced matrix

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{cc} B_2 & B_3 \end{array} \\ \begin{bmatrix} 8 & -2 \\ 0 & 6 \\ -6 & -4 \end{bmatrix} \end{array}$$

Here, the average of the pay-offs due to strategies A_1 and A_2 of firm A, i.e. $\left\{\frac{8+0}{2}, \frac{-2+6}{2}\right\} = \{4, 2\}$ dominates the pay-off due to A_3. So we get a new reduced 2x2 pay-off matrix.

Firm B's strategy

$$\begin{array}{c|cc} & B_2 & B_3 \\ \hline \end{array}$$

Firm A's strategy
$$\begin{array}{c} A_1 \\ A_2 \end{array}\begin{bmatrix} 8 & -2 \\ 0 & 6 \end{bmatrix}$$

We have a = 8, b = -2, c = 0 and d = 6.

$$p = \frac{d-c}{(a+d)-(b+c)}$$
$$= \frac{6-0}{(6+8)-(-2+0)}$$
$$= \frac{6}{16}$$
$$= \frac{3}{8}$$
$$1-p = 1-\frac{3}{8} = \frac{5}{8}$$
$$r = \frac{d-b}{(a+d)-(b+c)}$$
$$= \frac{6-(-2)}{16}$$
$$= \frac{8}{16}$$
$$= \frac{1}{2}$$
$$1-r = 1-\frac{1}{2} = \frac{1}{2}$$

Value of the game:

$$V = \frac{ad-bc}{(a+d)-(b+c)}$$
$$= \frac{6x8-0x(-2)}{16}$$
$$= \frac{48}{16} = 3$$

So the optimal strategies are

$$A = \left\{\frac{3}{8}, \frac{5}{8}, 0, 0, 0\right\} \text{ and } B = \left\{0, \frac{1}{2}, \frac{1}{2}, 0, 0\right\}.$$

The value of the game = 3. Thus the game is favourable to firm A.

Problem 3:

For the game with the following pay-off matrix, determine the saddle point

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{cccc} I & II & III & IV \end{array} \\ \text{Player A} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 2 & -1 & 0 & -3 \\ 1 & 0 & 3 & 2 \\ -3 & -2 & -1 & 4 \end{bmatrix} \end{array}$$

Solution:

$$\begin{array}{cc} & \textit{Column II} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{array}{|c|} -1 \\ 0 \\ -2 \end{array} \end{array} \qquad \begin{array}{c} \textit{Column III} \\ \begin{array}{|c|} 0 \\ 3 \\ -1 \end{array} \end{array} \qquad \begin{array}{c} 0 > -1 \\ 3 > 0 \\ -1 > -2 \end{array}$$

The choice is with the player B. He has to choose between strategies II and III. He will lose more in strategy III than in strategy II, irrespective of what strategy is followed by A. So he will drop strategy III and retain strategy II. Now the given game reduces to the following game.

$$\begin{array}{cc} & \begin{array}{ccc} I & II & IV \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 2 & -1 & -3 \\ 1 & 0 & 2 \\ -3 & -2 & 4 \end{bmatrix} \end{array}$$

Consider the rows and columns of this matrix.

Row minimum:

I Row	:	-3	
II Row	:	0	Maximum of {-3, 0, -3} = 0
III Row	:	-3	

Column maximum:

I Column	:	2	
II Column	:	0	Minimum of {2, 0, 4} = 0
III Column	:	4	

We see that

Maximum of row minimum = Minimum of column maximum = 0.

So, a saddle point exists for the given game and the value of the game is 0.

Interpretation:

No player gains and no player loses. i.e., The game is not favourable to any player. i.e. It is a fair game.

Problem 4:

Solve the game

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 4 & 8 & 6 \\ 6 & 2 & 10 \\ 4 & 5 & 7 \end{bmatrix} \end{array}$$

Solution:

First consider the minimum of each row.

Row	Minimum
1	4
2	2
3	4

Maximum of {4, 2, 4} = 4

Next, consider the maximum of each column.

Column	Maximum
1	6
2	8
3	10

Minimum of {6, 8, 10} = 6

Since Maximum of { Row Minima} and Minimum of { Column Maxima } are different, it follows that the given game has no saddle point.

Denote the strategies of player A by A_1, A_2, A_3. Denote the strategies of player B by B_1, B_2, B_3.

Compare the first and third columns of the given matrix.

$$\begin{array}{cc} B_1 & B_3 \\ \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix} & \begin{bmatrix} 6 \\ 10 \\ 7 \end{bmatrix} \end{array}$$

The pay-offs in B_3 are greater than or equal to the corresponding pay-offs in B_1. The player B has to make a choice between his strategies 1 and 3. He will lose more if he follows strategy 3 rather than strategy 1. Therefore he will give up strategy 3 and retain strategy 1. Consequently, the given game is transformed into the following game:

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 4 & 5 \end{bmatrix} \end{array}$$

Compare the first and third rows of the above matrix.

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_3 \end{array} & \begin{bmatrix} 4 & 8 \\ 4 & 5 \end{bmatrix} \end{array}$$

The pay-offs in A_1 are greater than or equal to the corresponding pay-offs in A_3. The player A has to make a choice between his strategies 1 and 3. He will gain more if he follows strategy 1 rather than strategy 3. Therefore he will retain strategy 1 and give up strategy 3. Now the given game is transformed into the following game.

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 4 & 8 \\ 6 & 2 \end{bmatrix} \end{array}$$

It is a 2x2 game. Consider the row minima.

Row	Minimum
1	4
2	2

Maximum of {4, 2} = 4

Next, consider the maximum of each column.

Column	Maximum
1	6
2	8

Minimum of {6, 8} = 6

Maximum {row minima} and Minimum {column maxima } are not equal

Therefore, the reduced game has no saddle point. So, it is a mixed game

Take $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 6 & 2 \end{bmatrix}$. We have a = 4, b = 8, c = 6 and d = 2.

The probability that player A will use his first strategy is p. This is calculated as

$$p = \frac{d-c}{(a+d)-(b+c)}$$
$$= \frac{2-6}{(4+2)-(8+6)}$$
$$= \frac{-4}{6-14}$$
$$= \frac{-4}{-8} = \frac{1}{2}$$

The probability that player B will use his first strategy is r. This is calculated as

$$r = \frac{d-b}{(a+d)-(b+c)}$$
$$= \frac{2-8}{-8}$$
$$= \frac{-6}{-8}$$
$$= \frac{3}{4}$$

Value of the game is V. This is calculated as

$$V = \frac{ad-bc}{(a+d)-(b+c)}$$
$$= \frac{4x2-8x6}{-8}$$
$$= \frac{8-48}{-8}$$
$$= \frac{-40}{-8} = 5$$

Interpretation

Out of 3 trials, player A will use strategy 1 once and strategy 2 once. Out of 4 trials, player B will use strategy 1 thrice and strategy 2 once. The game is favourable to player A.

Problem 5: Dividing a game into sub-games

Solve the game with the following pay-off matrix.

$$\text{Player A} \quad \begin{array}{c} \\ I \\ II \\ III \end{array} \begin{array}{c} \text{Player B} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} -4 & 6 & 3 \\ -3 & 3 & 4 \\ 2 & -3 & 4 \end{bmatrix} \end{array}$$

Solution:

First, consider the row mimima.

Row	Minimum
1	-4
2	-3
3	-3

Maximum of {-4, -3, -3} = -3

Next, consider the column maxima.

Column	Maximum
1	2
2	6
3	4

Minimum of {2, 6, 4} = 2

We see that Maximum of { row minima} $\neq$ Minimum of { column maxima}.

So the game has no saddle point. Hence it is a mixed game. Compare the first and third columns.

$$\begin{array}{ccc} \textit{I Column} & \textit{III Column} & \\ \begin{vmatrix} -4 \\ -3 \\ 2 \end{vmatrix} & \begin{vmatrix} 3 \\ 4 \\ 4 \end{vmatrix} & \begin{array}{r} -4 \le 3 \\ -3 \le 4 \\ 2 \le 4 \end{array} \end{array}$$

We assert that Player B will retain the first strategy and give up the third strategy. We get the following reduced matrix.

$$\begin{bmatrix} -4 & 6 \\ -3 & 3 \\ 2 & -3 \end{bmatrix}$$

We check that it is a game with no saddle point.

Sub games

Let us consider the 2x2 sub games. They are:

$$\begin{bmatrix} -4 & 6 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} -3 & 3 \\ 2 & -3 \end{bmatrix}$$

First, take the sub game

$$\begin{bmatrix} -4 & 6 \\ -3 & 3 \end{bmatrix}$$

Compare the first and second columns. We see that $-4 \le 6$ and $-3 \le 3$. Therefore, the game reduces to $\begin{bmatrix} -4 \\ -3 \end{bmatrix}$. Since $-4 < -3$, it further reduces to –3.

Next, consider the sub game

$$\begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix}$$

We see that it is a game with no saddle point. Take a = -4, b = 6, c = 2, d = -3. Then the value of the game is

$$\begin{aligned} V &= \frac{ad - bc}{(a+d)-(b+c)} \\ &= \frac{(-4)(-3)-(6)(2)}{(-4+3)-(6+2)} \\ &= 0 \end{aligned}$$

Next, take the sub game $\begin{bmatrix} -3 & 3 \\ 2 & -3 \end{bmatrix}$. In this case we have a = -3, b = 3, c = 2 and d = -3. The value of the game is obtained as

$$\begin{aligned} V &= \frac{ad - bc}{(a+d)-(b+c)} \\ &= \frac{(-3)(-3)-(3)(2)}{(-3-3)-(3+2)} \\ &= \frac{9-6}{-6-5} = -\frac{3}{11} \end{aligned}$$

Let us tabulate the results as follows:

Sub game	Value
$\begin{bmatrix} -4 & 6 \\ -3 & 3 \end{bmatrix}$	-3
$\begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix}$	0
$\begin{bmatrix} -3 & 3 \\ 2 & -3 \end{bmatrix}$	$-\frac{3}{11}$

The value of 0 will be preferred by the player A. For this value, the first and third strategies of A correspond while the first and second strategies of the player B correspond to the value 0 of the game. So it is a fair game.

QUESTIONS

1. Explain the principle of dominance in the theory of games.
2. Explain how a game can be solved through sub games.
3. Solve the following game by the principle of dominance:

Player B

Strategies

		I	*II*	*III*	*IV*
Player A Strategies	1	8	10	9	14
	2	10	11	8	12
	3	13	12	14	13

Answer: V = 12

4. Solve the game by the principle of dominance:

$$\begin{bmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 2 & 6 \end{bmatrix}$$

Answer: V = 4

5. Solve the game with the following pay-off matrix

$$\begin{bmatrix} 6 & 3 & -1 & 0 & -3 \\ 3 & 2 & -4 & 2 & -1 \end{bmatrix}$$

Answer **:** $p = \frac{3}{5},\ r = \frac{2}{5},\ V = -\frac{11}{5}$

6. Solve the game

$$\begin{bmatrix} 8 & 7 & 6 & -1 & 2 \\ 12 & 10 & 12 & 0 & 4 \\ 14 & 6 & 8 & 14 & 16 \end{bmatrix}$$

Answer **:** $p = \frac{4}{9},\ r = \frac{7}{9},\ V = \frac{70}{9}$

LESSON 5

GRAPHICAL SOLUTION OF A 2x2 GAME WITH NO SADDLE POINT

LESSON OUTLINE

- The principle of graphical solution
- Numerical example

LEARNING OBJECTIVES

After reading this lesson you should be able to

- understand the principle of graphical solution
- derive the equations involving probability and expected value
- solve numerical problems

Example: Consider the game with the following pay-off matrix.

Player B

$$\text{Player A} \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}$$

First consider the row minima.

Row	Minimum
1	2
2	1

Maximum of {2, 1} = 2.

Next, consider the column maxima.

Column	Maximum
1	4
2	5

Minimum of {4, 5} = 4.

We see that Maximum { row minima} $\neq$ Minimum { column maxima }

So, the game has no saddle point. It is a mixed game.

Equations involving probability and expected value:

Let p be the probability that player A will use his first strategy.

Then the probability that A will use his second strategy is 1-p.

Let E be the expected value of pay-off to player A.

When B uses his first strategy

The expected value of pay-off to player A is given by

$$\begin{aligned} E &= 2p + 4(1-p) \\ &= 2p + 4 - 4p \qquad (1) \\ &= 4 - 2p \end{aligned}$$

When B uses his second strategy

The expected value of pay-off to player A is given by

$$\begin{aligned} E &= 5p + 1(1-p) \\ &= 5p + 1 - p \qquad (2) \\ &= 4p + 1 \end{aligned}$$

Consider equations (1) and (2). For plotting the two equations on a graph sheet, get some points on them as follows:

$E = -2p+4$

p	0	1	0.5
E	4	2	3

$E = 4p+1$

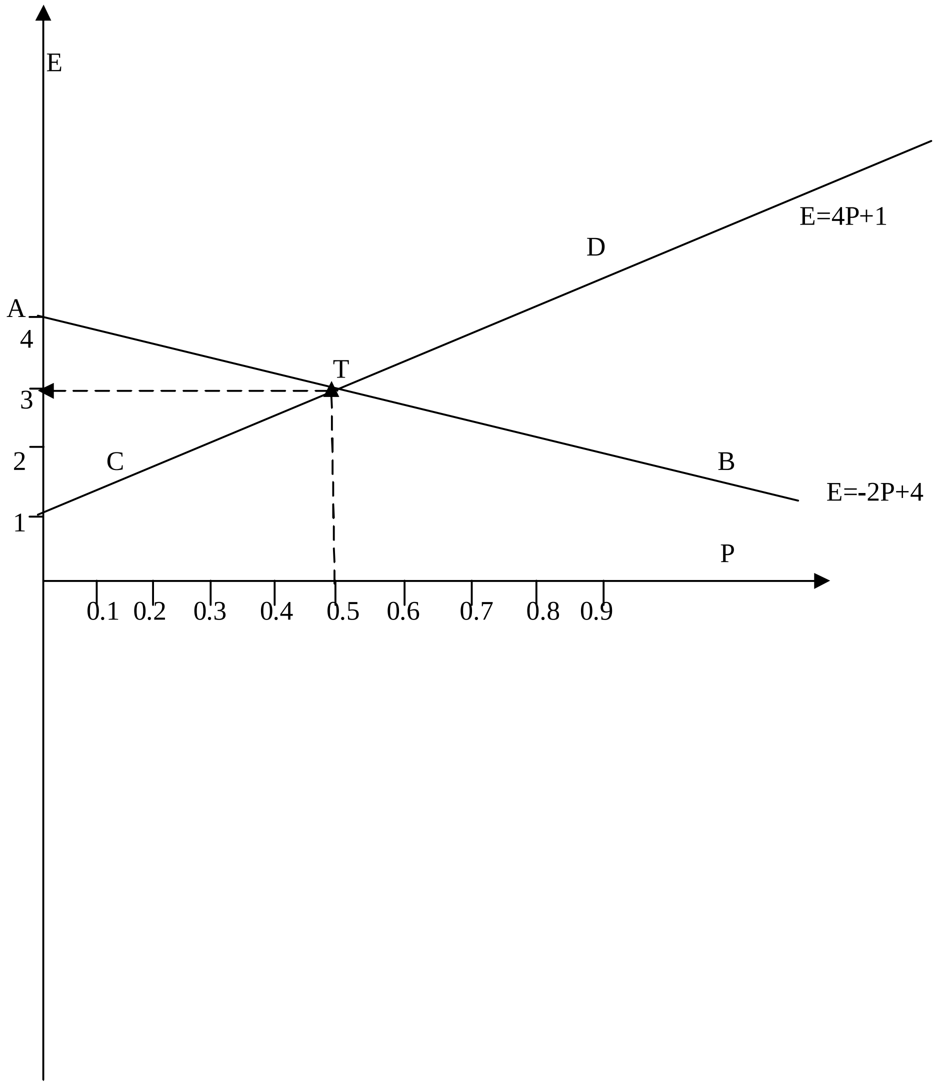
p	0	1	0.5
E	1	5	3

Graphical solution:

Procedure:

Take probability and expected value along two rectangular axes in a graph sheet. Draw two straight lines given by the two equations (1) and (2). Determine the point of intersection of the two straight lines in the graph. This will give the common solution of the two equations (1) and (2). Thus we would obtain the value of the game.

Represent the two equations by the two straight lines AB and CD on the graph sheet. Take the point of intersection of AB and CD as T. For this point, we have $p = 0.5$ and $E = 3$. Therefore, the value V of the game is 3.



Problem 1:

Solve the following game by graphical method.

Player B

$$\text{Player A} \begin{bmatrix} -18 & 2 \\ 6 & -4 \end{bmatrix}$$

Solution:

First consider the row minima.

Row	Minimum
1	- 18
2	- 4

$$\text{Maximum of } \{-18, -4\} = -4.$$

Next, consider the column maxima.

Column	Maximum
1	6
2	2

$$\text{Minimum of } \{6, 2\} = 2.$$

We see that Maximum { row minima} $\neq$ Minimum { column maxima }

So, the game has no saddle point. It is a mixed game.

Let p be the probability that player A will use his first strategy.

Then the probability that A will use his second strategy is 1-p.

When B uses his first strategy

The expected value of pay-off to player A is given by

$$\begin{aligned} E &= -18p + 6(1-p) \\ &= -18p + 6 - 6p \qquad \text{(I)} \\ &= -24p + 6 \end{aligned}$$

When B uses his second strategy

The expected value of pay-off to player A is given by

$$\begin{aligned} E &= 2p - 4(1-p) \\ &= 2p - 4 + 4p \qquad \text{(II)} \\ &= 6p - 4 \end{aligned}$$

Consider equations (I) and (II). For plotting the two equations on a graph sheet, get some points on them as follows:

E = -24 p + 6

p	0	1	0.5
E	6	-18	-6

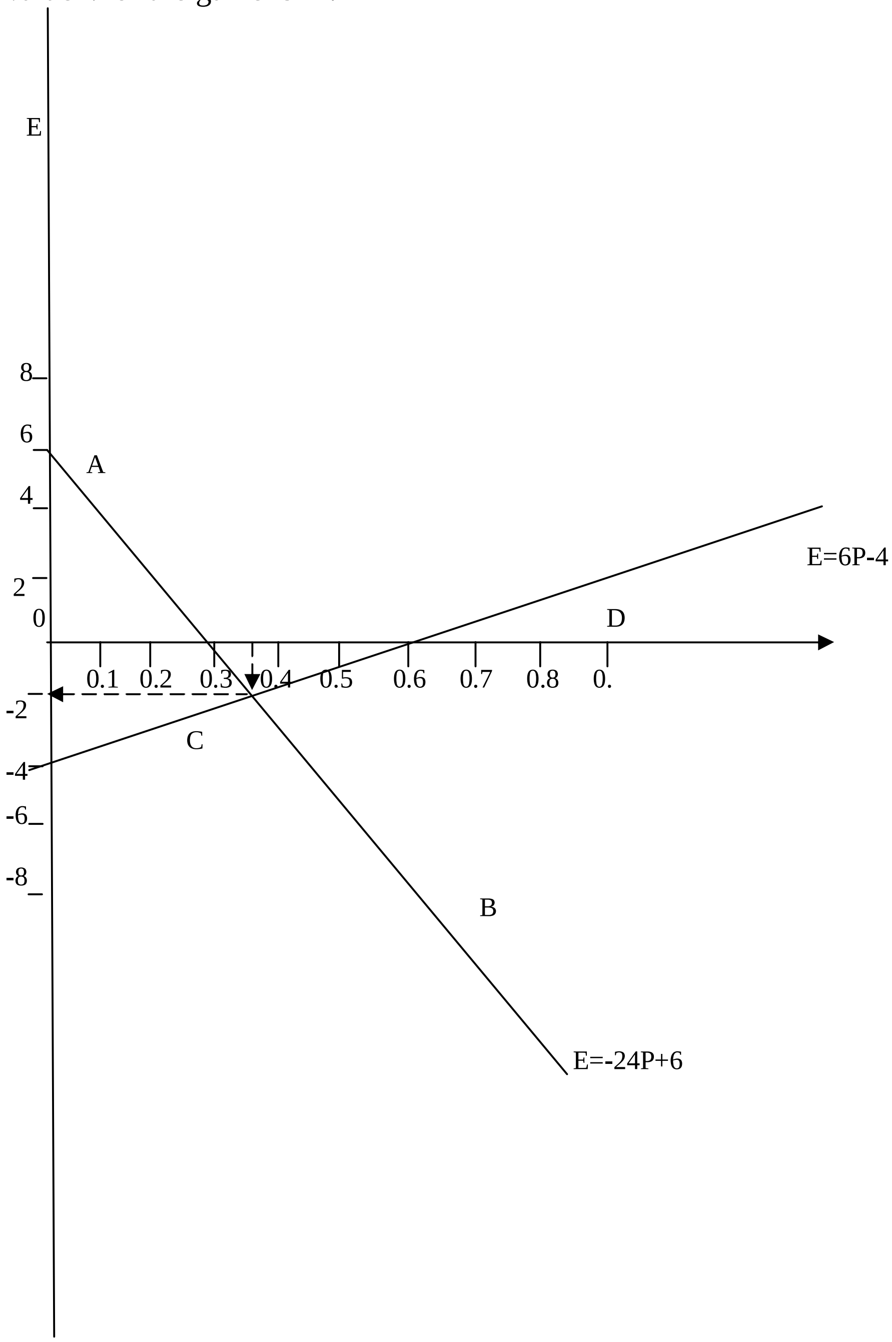
E = 6p-4

p	0	1	0.5
E	-4	2	-1

Graphical solution:

Take probability and expected value along two rectangular axes in a graph sheet. Draw two straight lines given by the two equations (1) and (2). Determine the point of intersection of the two straight lines in the graph. This will provide the common solution of the two equations (1) and (2). Thus we would get the value of the game.

Represent the two equations by the two straight lines AB and CD on the graph sheet. Take the point of intersection of AB and CD as T. For this point, we have p = $\frac{1}{3}$ and E = -2. Therefore, the value V of the game is -2.



QUESTIONS

1. Explain the method of graphical solution of a 2x2 game.

2. Obtain the graphical solution of the game

$$\begin{bmatrix} 10 & 6 \\ 8 & 12 \end{bmatrix}$$

Answer: $p = \frac{1}{2}$, $V = 9$

3. Graphically solve the game

$$\begin{bmatrix} 4 & 10 \\ 8 & 6 \end{bmatrix}$$

Answer: $p = \frac{1}{4}$, $V = 7$

4. Find the graphical solution of the game

$$\begin{bmatrix} -12 & 12 \\ 2 & -6 \end{bmatrix}$$

Answer: $p = \frac{1}{4}$, $V = -\frac{3}{2}$

5. Obtain the graphical solution of the game

$$\begin{bmatrix} 10 & 6 \\ 8 & 12 \end{bmatrix}$$

Answer: $p = \frac{1}{2}$, $V = 9$

6. Graphically solve the game

$$\begin{bmatrix} -3 & -5 \\ -5 & 1 \end{bmatrix}$$

Answer: $p = \frac{3}{4}$, $V = -\frac{7}{2}$

LESSON 6

2 x n ZERO-SUM GAMES

LESSON OUTLINE

- A 2 x n zero-sum game
- Method of solution
- Sub game approach and graphical method
- Numerical example

LEARNING OBJECTIVES

After reading this lesson you should be able to

- understand the concept of a 2 x n zero-sum game
- solve numerical problems

The concept of a 2 x n zero-sum game

When the first player A has exactly two strategies and the second player B has n (where n is three or more) strategies, there results a 2 x n game. It is also called a rectangular game. Since A has two strategies only, he cannot try to give up any one of them. However, since B has many strategies, he can make out some choice among them. He can retain some of the advantageous strategies and discard some disadvantageous strategies. The intention of B is to give as minimum payoff to A as possible. In other words, B will always try to minimize the loss to himself. Therefore, if some strategies are available to B by which he can minimize the payoff to A, then B will retain such strategies and give such strategies by which the payoff will be very high to A.

Approaches for 2 x n zero-sum game

There are two approaches for such games: (1) Sub game approach and (2) Graphical approach.

Sub game approach

The given 2 x n game is divided into 2 x 2 sub games. For this purpose, consider all possible 2 x 2 sub matrices of the payoff matrix of the given game. Solve each sub game and have a list of the values of each sub game. Since B can make out a choice of his strategies, he will discard such of those sub games which result in more payoff to A. On the basis of this consideration, in the long run, he will retain two strategies only and give up the other strategies.

Problem

Solve the following game

Player B

$$\text{Player A} \begin{bmatrix} 8 & -2 & -6 & 9 \\ 3 & 5 & 10 & 2 \end{bmatrix}$$

Solution:

Let us consider all possible 2x2 sub games of the given game. We have the following sub games:

1. $\begin{bmatrix} 8 & -2 \\ 3 & 5 \end{bmatrix}$
2. $\begin{bmatrix} 8 & -6 \\ 3 & 10 \end{bmatrix}$
3. $\begin{bmatrix} 8 & 9 \\ 3 & 2 \end{bmatrix}$
4. $\begin{bmatrix} -2 & -6 \\ 5 & 10 \end{bmatrix}$
5. $\begin{bmatrix} -2 & 9 \\ 5 & 2 \end{bmatrix}$
6. $\begin{bmatrix} -6 & 9 \\ 10 & 2 \end{bmatrix}$

Let E be the expected value of the pay off to player A. Let p be the probability that player A will use his first strategy. Then the probability that he will use his second strategy is 1-p. We form the equations for E in all the sub games as follows:

Sub game (1)

Equation 1: $E = 8p + 3(1-p) = 5p + 3$

Equation 2: $E = -2p + 5(1-p) = -7p + 5$

Sub game (2)

Equation 1: $E = 8p + 3(1-p) = 5p + 3$

Equation 2: $E = -6p + 10(1-p) = -16p + 10$

Sub game (3)

Equation 1: $E = 8p + 3(1-p) = 5p + 3$

Equation 2: $E = 9p + 2(1-p) = 7p + 2$

Sub game (4)

Equation 1: $E = -2p + 5(1-p) = -7p + 5$

Equation 2: $E = -6p + 10(1-p) = -16p + 10$

Sub game (5)

Equation 1: $E = -2p + 5(1-p) = -7p + 5$

Equation 2: $E = 9p + 2(1-p) = 7p + 2$

Sub game (6)

Equation 1: $E = -6p + 10(1-p) = -16p + 10$

Equation 2: $E = 9p + 2(1-p) = 7p + 2$

Solve the equations for each sub game. Let us tabulate the results for the various sub games. We have the following:

Sub game	p	Expected value E
1	$\frac{1}{6}$	$\frac{23}{6}$
2	$\frac{1}{3}$	$\frac{14}{3}$
3	$\frac{1}{2}$	$\frac{11}{2}$
4	$\frac{5}{9}$	$\frac{10}{9}$
5	$\frac{3}{14}$	$\frac{7}{2}$
6	$\frac{8}{23}$	$\frac{102}{23}$

Interpretation:

Since player A has only 2 strategies, he cannot make any choice on the strategies. On the other hand, player B has 4 strategies. Therefore he can retain any 2 strategies and give up the other 2 strategies. This he will do in such a way that the pay-off to player A is at the minimum. The pay-off to A is the minimum in the case of sub game 4. i.e., the sub game with the matrix $\begin{bmatrix} -2 & -6 \\ 5 & 10 \end{bmatrix}$.

Therefore, in the long run, player B will retain his strategies 2 and 3 and give up his strategies 1 and 4. In that case, the probability that A will use his first strategy is p = $\frac{5}{9}$ and the probability that he will use his second strategy is 1-p = $\frac{4}{9}$. i.e., Out of a total of 9 trials, he will use his first strategy five times and the second strategy four times. The value of the game is $\frac{10}{9}$. The positive sign of V shows that the game is favourable to player A.

GRAPHICAL SOLUTION:

Now we consider the graphical method of solution to the given game.

Draw two vertical lines MN and RS. Note that they are parallel to each other. Draw UV perpendicular to MN as well as RS. Take U as the origin on the line MN. Take V as the origin on the line RS.

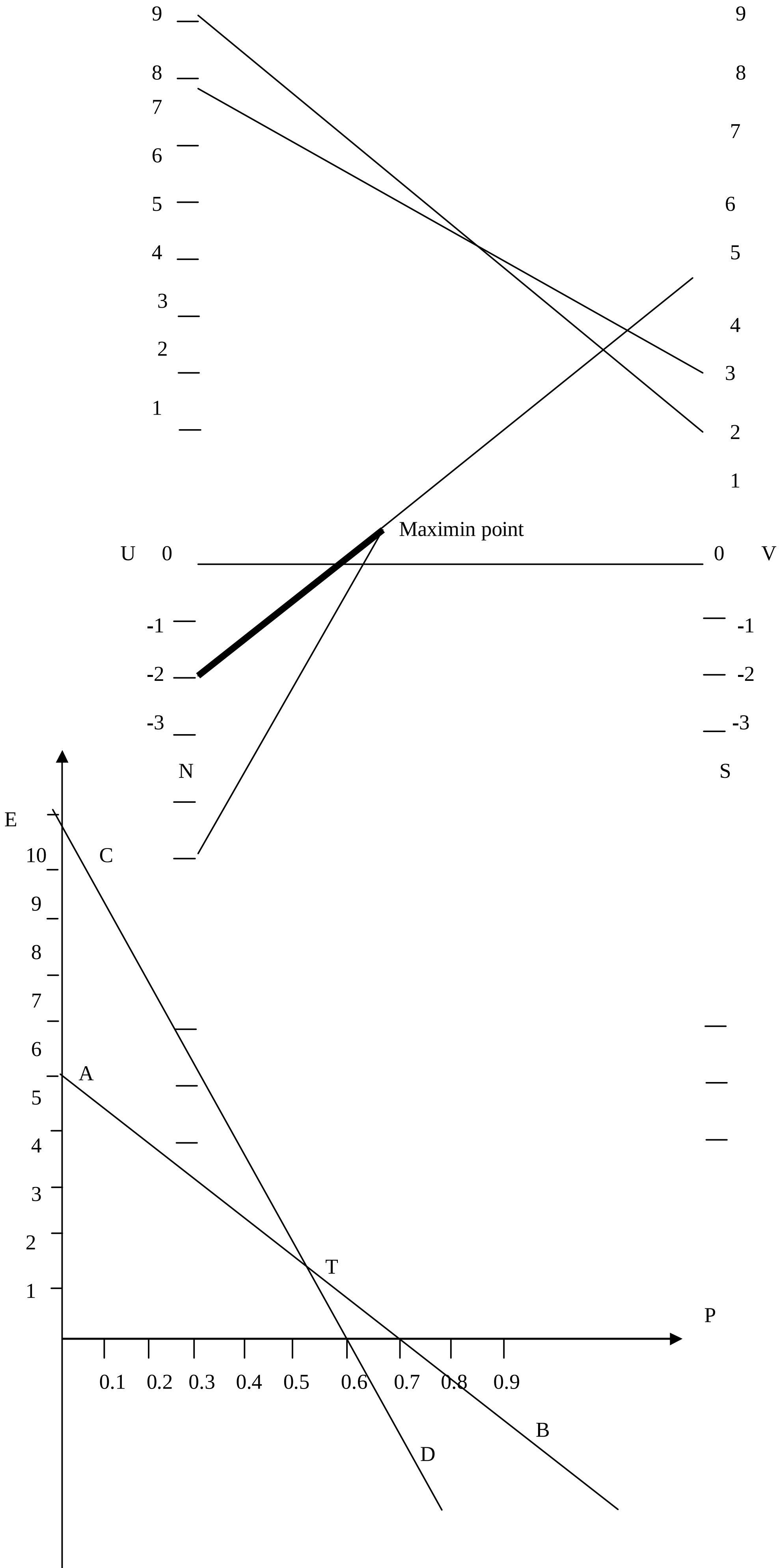
Mark units on MN and RS with equal scale. The units on the two lines MN and RS are taken as the payoff numbers. The payoffs in the first row of the given matrix are taken along the line MN while the payoffs in the second row are taken along the line RS.

We have to plot the following points: (8, 3), (-2, 5), (-6, 10), (9, 2). The points 8, -2, -6, 9 are marked on MN. The points 3, 5, 10, 2 are marked on RS.

Join a point on MN with the corresponding point on RS by a straight line. For example, join the point 8 on MN with the point 3 on RS. We have 4 such straight lines. They represent the 4 moves of the second player. They intersect in 6 points. Take the lowermost point of intersection of the straight lines. It is called the **Maximin point**. With the help of this point, identify the optimal strategies for the second player. This point corresponds to the points –2 and –6 on MN and 5 and 10 on RS. They correspond to the sub game with the matrix $\begin{bmatrix} -2 & -6 \\ 5 & 10 \end{bmatrix}$.

The points –2 and –6 on MN correspond to the second and third strategies of the second player. Therefore, the graphical method implies that, in the long run, the second player will retain his strategies 2 and 3 and give up his strategies 1 and 4.

We graphically solve the sub game with the above matrix. We have to solve the two equations E = -7 p + 5 and E = - 16 p + 10. Represent the two equations by two straight lines AB and CD on the graph sheet. Take the point of intersection of AB and CD as T. For this point, we have p = $\frac{5}{9}$ and E = $\frac{10}{9}$. Therefore, the value V of the game is $\frac{10}{9}$. We see that the probability that first player will use his first strategy is p = $\frac{5}{9}$ and the probability that he will use his second strategy is 1-p = $\frac{4}{9}$.


Maximin point
U
0
V
N
S
E
C
A
T
P
B
D
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9

E=-7P+5

E=-16P+10

E = - 7p+5

p	0	1	0.5
E	5	- 2	1.5

E = - 16p+10

p	0	1	0.5
E	10	- 6	2

QUESTIONS

1. Explain a 2 x n zero-sum game.
2. Describe the method of solution of a 2 x n zero-sum game.
3. Solve the following game:

   Player B

$$\text{Player A} \begin{bmatrix} 10 & 2 & 6 \\ 1 & 5 & 8 \end{bmatrix}$$

Answer: $p = \frac{1}{3}$, $V = 4$

LESSON 7

m x 2 ZERO-SUM GAMES

LESSON OUTLINE

- An m x 2 zero-sum game
- Method of solution
- Sub game approach and graphical method
- Numerical example

LEARNING OBJECTIVES

After reading this lesson you should be able to

- understand the concept of an m x 2 zero-sum game
- solve numerical problems

The concept of an m x 2 zero-sum game

When the second player B has exactly two strategies and the first player A has m (where m is three or more) strategies, there results an m x 2 game. It is also called a rectangular game. Since B has two strategies only, he will find it difficult to discard any one of them. However, since A has

more strategies, he will be in a position to make out some choice among them. He can retain some of the most advantageous strategies and give up some other strategies. The motive of A is to get as maximum payoff as possible. Therefore, if some strategies are available to A by which he can get more payoff to himself, then he will retain such strategies and discard some other strategies which result in relatively less payoff.

Approaches for m x 2 zero-sum game

There are two approaches for such games: (1) Sub game approach and (2) Graphical approach.

Sub game approach

The given m x 2 game is divided into 2 x 2 sub games. For this purpose, consider all possible 2 x 2 sub matrices of the payoff matrix of the given game. Solve each sub game and have a list of the values of each sub game. Since A can make out a choice of his strategies, he will be interested in such of those sub games which result in more payoff to himself. On the basis of this consideration, in the long run, he will retain two strategies only and give up the other strategies.

Problem

Solve the following game:

Player B

Strategies

Player A Strategies		*I*	*II*
	1	5	8
	2	–2	10
	3	12	4
	4	6	5

Solution:

Let us consider all possible 2x2 sub games of the given game. We have the following sub games:

7. $\begin{bmatrix} 5 & 8 \\ -2 & 10 \end{bmatrix}$

8. $\begin{bmatrix} 5 & 8 \\ 12 & 4 \end{bmatrix}$

9. $\begin{bmatrix} 5 & 8 \\ 6 & 5 \end{bmatrix}$

10. $\begin{bmatrix} -2 & 10 \\ 12 & 4 \end{bmatrix}$

11. $\begin{bmatrix} -2 & 10 \\ 6 & 5 \end{bmatrix}$

12. $\begin{bmatrix} 12 & 4 \\ 6 & 5 \end{bmatrix}$

Let E be the expected value of the payoff to player A. i.e., the loss to player B. Let r be the probability that player B will use his first strategy. Then the probability that he will use his second strategy is 1-r. We form the equations for E in all the sub games as follows:

Sub game (1)

Equation 1: $E = 5r + 8(1-r) = -3r + 8$

Equation 2: $E = -2r + 10(1-r) = -12r + 10$

Sub game (2)

Equation 1: $E = 5r + 8(1-r) = -3r + 8$

Equation 2: $E = 12r + 4(1-r) = 8r + 4$

Sub game (3)

Equation 1: $E = 5r + 8(1-r) = -3r + 8$

Equation 2: $E = 6r + 5(1-r) = r + 5$

Sub game (4)

Equation 1: $E = -2r + 10(1-r) = -12r + 10$

Equation 2: $E = 12r + 4(1-r) = 8r + 4$

Sub game (5)

Equation 1: $E = -2r + 10(1-r) = -12r + 10$

Equation 2: $E = 6r + 5(1-r) = r + 5$

Sub game (6)

Equation 1: $E = 12r + 4(1-r) = 8r + 4$

Equation 2: $E = 6r + 5(1-r) = r + 5$

Solve the equations for each 2x2 sub game. Let us tabulate the results for the various sub games. We have the following:

Sub game	R	Expected value E
1	$\frac{2}{9}$	$\frac{22}{3}$

2	$\frac{4}{11}$	$\frac{76}{11}$
3	$\frac{3}{4}$	$\frac{23}{4}$
4	$\frac{3}{10}$	$\frac{32}{5}$
5	$\frac{5}{13}$	$\frac{70}{13}$
6	$\frac{1}{7}$	$\frac{36}{7}$

Interpretation:

Since player B has only 2 strategies, he cannot make any choice on his strategies. On the other hand, player A has 4 strategies and so he can retain any 2 strategies and give up the other 2 strategies. Since the choice is with A, he will try to maximize the payoff to himself. The pay-off to A is the maximum in the case of sub game 1. i.e., the sub game with the matrix $\begin{bmatrix} 5 & 8 \\ -2 & 10 \end{bmatrix}$.

Therefore, player A will retain his strategies 1 and 2 and discard his strategies 3 and 4, in the long run. In that case, the probability that B will use his first strategy is r = $\frac{2}{9}$ and the probability that he will use his second strategy is 1-r = $\frac{7}{9}$. i.e., Out of a total of 9 trials, he will use his first strategy two times and the second strategy seven times.

The value of the game is $\frac{22}{3}$. The positive sign of V shows that the game is favourable to player A.

GRAPHICAL SOLUTION:

Now we consider the graphical method of solution to the given game.

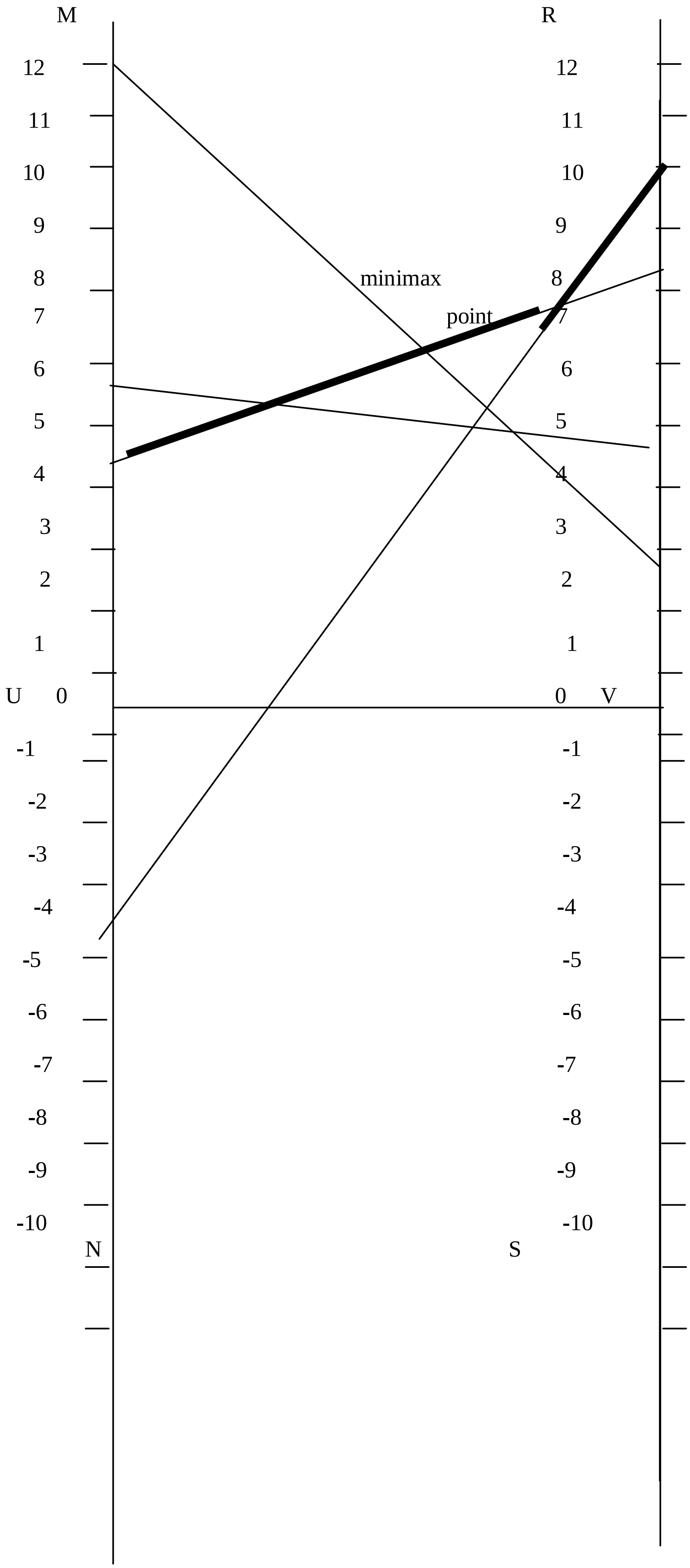
Draw two vertical lines MN and RS. Note that they are parallel to each other. Draw UV perpendicular to MN as well as RS. Take U as the origin on the line MN. Take V as the origin on the line RS.

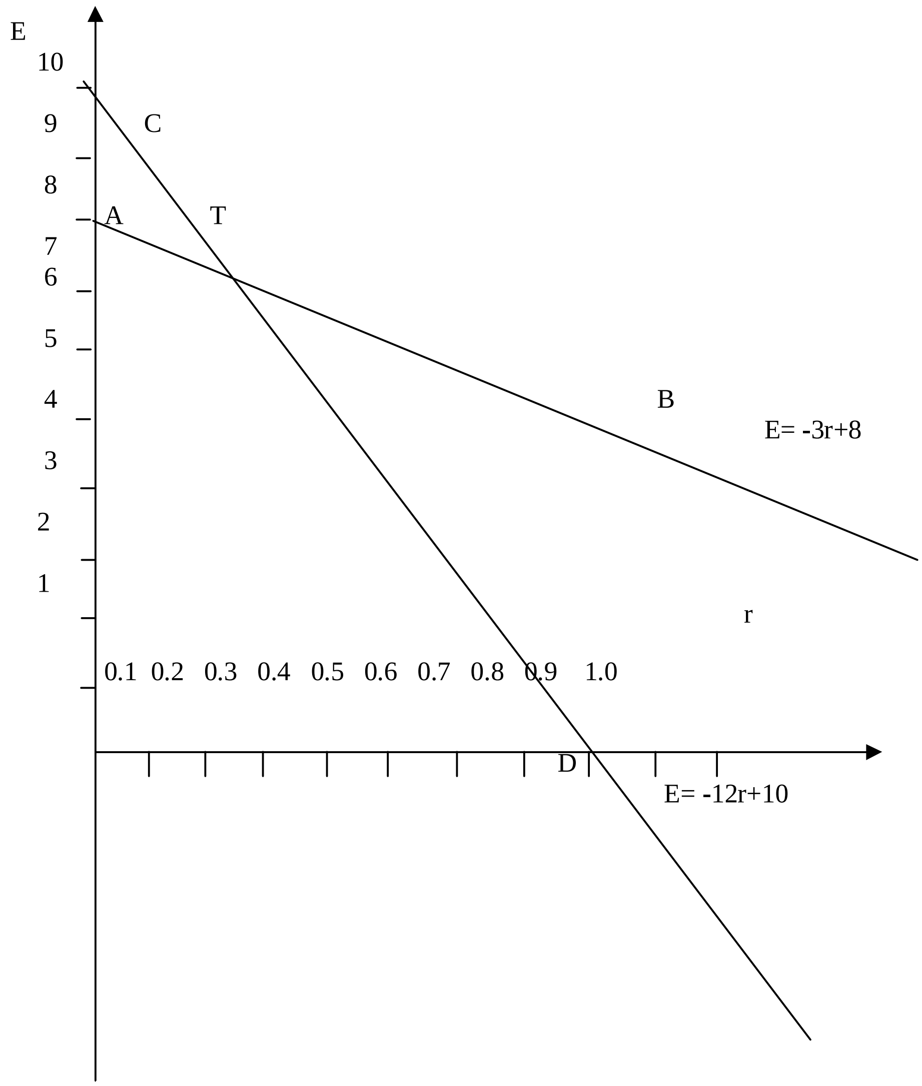
Mark units on MN and RS with equal scale. The units on the two lines MN and RS are taken as the payoff numbers. The payoffs in the first row of the given matrix are taken along the line MN while the payoffs in the second row are taken along the line RS.

We have to plot the following points: (5, 8), (-2, 10), (12, 4), (6, 5).The points 5, -2, 12, 6 are marked on MN. The points 8, 10, 4, 5 are marked on RS.

Join a point on MN with the corresponding point on RS by a straight line. For example, join the point 5 on MN with the point 8 on RS. We have 4 such straight lines. They represent the 4 moves of the first player. They intersect in 6 points. Take the uppermost point of intersection of the straight lines. It is called the **Minimax point**. With the help of this point, identify the optimal strategies for the first player. This point corresponds to the points 5 and -2 on MN and 8 and 10 on RS. They correspond to the sub game with the matrix $\begin{bmatrix} 5 & 8 \\ -2 & 10 \end{bmatrix}$. The points 5 and -2 on MN correspond to the first and second strategies of the first player. Therefore, the graphical method implies that the first player will retain his strategies 1 and 2 and give up his strategies 3 and 4, in the long run.

We graphically solve the sub game with the above matrix. We have to solve the two equations E = - 3 r + 8 and E = - 12 r + 10. Represent the two equations by two straight lines AB and CD on the graph sheet. Take the point of intersection of AB and CD as T. For this point, we have r = $\frac{2}{9}$ and E = $\frac{22}{3}$. Therefore, the value V of the game is $\frac{22}{3}$. We see that the probability that the second player will use his first strategy is r = $\frac{2}{9}$ and the probability that he will use his second strategy is 1-r = $\frac{7}{9}$.


M
R
12
11
10
9
8
7
6
5
4
3
2
1
U
0
V
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
N
S
minimax
point



$E = -3r+8$

p	0	1	0.5
E	8	5	6.5

$E = -12r+10$

p	0	1	0.5
E	10	- 2	4