Sub :Advanced Calculus and Numerical Methods Sub-Code : 18MAT21

Module 5

NUMERICAL METHOD

CONTENTS:

- Finite difference:
  Forward and Backward difference
- Newtons forward and backward interpolation formula
- Newtons divided difference formula
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NUMERICAL METHODS

Finite Differences

Let y = f(x) be represented by a table

x :	x_0	x_1	x_2	x_3	….	x_n
y :	y_0	y_1	y_2	y_3	…	y_n

where x_0, x_1,x_2….x_n are equidistant. ($x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \ldots = x_n - x_{n-1} = h$)

Forward difference operator (Δ) The first forward difference is defined as fpllows:

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta y_r = y_{r+1} - y_r, \; r = 0,1,2,\ldots, n-1$$

$$\left.\begin{array}{l} \Delta y_0 = y_1 - y_0 \\ \Delta y_1 = y_2 - y_1 \\ . \\ . \\ \Delta y_{n-1} = y_n - y_{n-1} \end{array}\right\} \text{first forward differences}$$

The difference of the first difference are called the second differences,they are symbolically denoted as

$$\text{Now } \Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0)$$

$$\Delta^2 y_0, \Delta^2 y_1, \Delta^2 y_2, \ldots, \text{are called the second differences}$$

$$= \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0)$$

$$= y_2 - 2y_1 + y_0$$

$$|||^{ly} \qquad \Delta^2 y_1 = y_3 - 2y_2 + y_1$$

$$\Delta^2 y_r = y_{r+2} - 2y_{r+1} + y_r$$

$$\text{Note}: \Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

$$\therefore \qquad \Delta^k y_r = y_{r+k} - {}^kC_1 y_{r+k-1} + {}^kC_2 \, y_{r+k-2} - + \ldots + (-1)\; {}^kC_r$$

The above forward Differences can be put in the following form called Difference Table

x	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_0$		
x_3	y_3		$\Delta^2 y_2$			
		Δy_3				
x_4	y_4					

$\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, ...$ are called the leading differences.

Ex: The following table gives a set of values of x and the corresponding values of y = f(x)

x :	10	15	20	25	30	35
y :	19.97	21.51	22.47	23.52	24.65	25.89

Form the difference table and find $\Delta f(10), \Delta^2 f(10), \Delta^3 f(20), \Delta^4 f(15)$.

x	Y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
10	19.97					
		1.54				
15	21.51		-0.58			
		0.96		0.67		
20	22.47		0.09		-0.68	

		1.05		-0.01		0.72
25	23.52		0.08		0.04	
		1.13		0.03		
30	24.65		0.11			
		1.24				
35	25.89					

$\Delta f(10) = 1.54,\ \Delta^2 f(10) = -0.58,\ \Delta^3 f(20) = 0.03,\ \Delta^4 f(15) = 0.04$

Note: The nth differences of a polynomial of n the degree are constant.

Backward difference operator (∇)

Let y = f(x) , then the backward difference is defined and denoted symbolically as

$\nabla f(x) = f(x) - f(x - h)$

i.e. $\nabla y_1 = y_1 - y_0 = \Delta y_0$

$\nabla y_2 = y_2 - y_1 = \Delta y_1$

$\nabla y_3 = y_3 - y_2 = \Delta y_2$

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$\nabla y_n = y_n - y_{n-1} = \Delta y_{n-1}$

$\therefore \quad \nabla y_r = y_r - y_{r-1} = \Delta y_{r-1}$

Note:

1. $\nabla f(x + h) = f(x + h) - f(x) = \Delta f(x)$
2. $\nabla^2 f(x + 2h) = \nabla(\nabla f(x + 2h))$

$= \nabla \{f(x + 2h) - f(x + h)\}$

$= \nabla f(x + 2h) - \nabla f(x + h)$

$= f(x + 2h) - f(x + h) - f(x + h) + f(x)$

$= f(x + 2h) - 2f(x + h) + f(x)$

$= \Delta^2 f(x)$

$|||^{ly}$ $\nabla^n f(x + nh) = \Delta^n f(x)$

Backward difference table

x	Y	∇y	$\nabla^2 y$	$\nabla^3 y$
X_0	y_0			
		∇y_1		
X_1	y_1		$\nabla^2 y_2$	
		∇y_2		$\nabla^3 y_3$
X_2	y_2		$\nabla^2 y_3$	
		∇y_3		
X_3	y_3			

1. Form the difference table for

X	40	50	60	70	80	90
Y	184	204	226	250	276	304

and find ∇y (30), $\nabla^2 y$ (70), $\nabla^5 y$ (90)

Soln:

x	Y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
40	184					
		20				
50	204		2			
		22		0		
60	226		2		0	
		24		0		0
70	250		2		0	

		26		0		
80	276		2			
		28				
90	304					

$\nabla y\ (80) = 26,\ \nabla^2 y\ (70) = 2,\ \nabla^5 y\ (90) = 0$

2. Given

X	0	1	2	3	4
f(x)	4	12	32	76	156

Construct the difference table and write the values of $\nabla f\ (4),\ \nabla^2 f\ (4),\ \nabla^3 f\ (3)$

x	Y	∇y	$\nabla^2 y$	$\nabla^3 y$
0	4			
		8		
1	12		12	
		20		12
2	32		24	
		44		12
3	76		36	
		80		
4	156			

3) Find the missing term from the table:

X	0	1	2	3	4
Y	1	3	9	-	81

Explain why the value obtained is different by putting x = 3 in 3^x.

Denoting the missing value as a, b, c ..etc. Construct a difference table and solve.

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1	2			
1	3	6	4		
2	9	a - 9	a - 15	a - 19	-4a + 124
3	a	81 - a	81 - a	-3a +105	
4	81				

Put $\Delta^4 y = 0$ (assuming f(x) its be a polynomial of degree 3)

i.e., $-4a + 124 = 0$

$a = 31$

Since we have assumed f(x) to be a polynomial of degree 3 which is not 3^x we obtained a different value.

4) Given $u_1 = 8$, $u_3 = 64$, $u_5 = 216$ find u_2 and u_4

X	u	Δu	$\Delta^2 u$	$\Delta^3 u$
x_1	8			
x_2	a	a - 8	-2a + 72	b + 3a - 200
x_3	64	64 - a	b + a - 128	-3b - a + 408
x_4	b	b - 64	-2b + 280	
x_5	216	216 -b		

We carryout upto the stage where we get two entries (Θ 2 unknowns) and equate each of those entries to zero. (Assuming) to be a polynomial of degree 2.

$b + 3a - 200 = 0$

$-3b - a + 408 = 0$ We get $a = 24$ $b = 128$

Interpolation:

Interpolation is the process of finding the intermediate values for a given set of values(x_0,y_0) ,(x_1,y_1),……………..(x_n, y_n) in the interval of the function y=f(x).The process of finding the value outside the interval(x_0,x_n) is called Extrapolation.

Newton-Gregory Forward Interpolation Formula

Given (x_0,y_0) ,(x_1,y_1),……………..(x_n, y_n) , for $y=\phi(x)$ such that$x_1=x_0+h, x_2=x_1+h$………………..$x_n=x_0+uh$, we wish to estimate the value of y corresponding to a desired value of x that lies near x_0 by using the following formula:

$$\varphi(x) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + ...$$

$$+ \frac{u(u-1)(u-2)...(u-n+1)}{n!}\Delta^n y_0$$

whichis called the Newton Gregory forward difference formula

Note :

1. Newton forward interpolation is generally used to interpolate the values of y near the beginning of a set of tabular values for a better accuracy,.

Problems:

1) The table gives the distances in nautical miles of the visible horizon for the given heights in feet above the earths surface.

x = height	100	150	200	250	300	350	400
y = distance	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when i) x = 120, ii) y =218

Solution:

X	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
100	10.63						
		2.40					
150	13.03		-0.39				
		2.01		0.15			
200	15.04		-0.24		-0.07		
		1.77		0.08		0.02	
250	16.81		-0.16		-0.05		0.02
		1.61		0.03		0.04	
300	18.42		-0.13		-0.01		
		1.48		0.02			
350	19.90		-0.11				
		1.37					
400	21.27						

Choose $x_0 = 100$

i) $x = 120, \ u = \frac{120-100}{50} = 0.4$

$$f(120) = 10.63 + \frac{0.4}{1!}(2.40) + \frac{(0.4)(0.4-1)}{2!}(-0.39)$$

$$+\frac{(0.4)(0.4-1)(0.4-2)}{3!}(0.15)$$

$$+\frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{4!}(-0.07)$$

$$+\frac{(0.4)(0.4-1)(0.4-2)(0.4-3)(0.4-4)}{5!}(0.02)$$

$$+\frac{(0.4)(0.4-1)(0.4-2)(0.4-3)(0.4-4)(0.4-5)}{6!}(0.02) = 11.649$$

ii) Let x = 218, $x_0 = 200$, $u = \frac{218-200}{50} = \frac{18}{50} = 0.36$

$$f(218) = 15.04 + 0.36(1.77) + \frac{0.36\,(-0.64)}{2}\,(-0.16)$$

$$+\frac{0.36(-0.64)\,(-1.64)}{6}(0.03) + \dots$$

$$= 15.7$$

2) Find the value of f(1.85).

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.7	5.474						
		0.575					
1.8	6.049		0.062				
		0.637		0.004			
1.9	6.686		0.066		0.004		
		0.703		0.008		-0.004	
2.0	7.389		0.074		0		0.004
		0.777		0.008		0	
2.1	8.166		0.082		0		
		0.859		0.008			
2.2	9.025		0.090				
		0.949					
23	9.974						

Choose $x_0 = 1.8,\ x = 1.85\ u = \frac{x - x_0}{h} = \frac{1.85 - 1.8}{0.1} = 0.5$

$$f(1.85) = 6.049 + (0.5)\,(0.637) + \frac{(0.5)(-0.5)}{2}(0.066)$$

$$+\frac{(0.5)(-0.5)\,(-1.5)}{6}\,(0.008)$$

$$= 6.049 + 0.3185 - 0.0008 + 0.0005$$

$$= 6.359$$

3) Given sin 45^o = 0.7071, sin 50^o = 0.7660, sin 55^o =0.8192, sin 60^o = 0.8660. Find sin 48^o.

x	Y	Δ	Δ^2	Δ^3
45	0.7071			
		0.589		
50	0.7660		-0.0057	
		0.0532		0.0007
55	0.8192		-0.0064	
		0.0468		
60	0.8660			

$$x = 48,\ x_0 = 45;\ h = 5\ u = \frac{x - x_0}{h} = 0.6$$

$$\sin\ 48^o = 0.7071 + (0.6)\ (0.0589)$$

$$+\frac{(0.6)(-0.4)}{2}\ (-0.0057)\ \ +\frac{(0.6)(-0.4)\ (-1.4)}{6}\ (0.0007) = 0.7431$$

4) From the following data find the number of students who have obtained ≤ 45 marks. Also find the number of students who have scored between 41 and 45 marks.

Marks	0 - 40	41 - 50	51 - 60	61 -70	71 - 80
No. of students	31	42	51	35	31

x	Y	Δ	Δ^2	Δ^3	Δ^4
40	31				

		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

$$f(45) = 31 + (0.5)\,(42) + \frac{(0.5)\,(-0.5)\,9}{2} + \frac{(0.5)\,(-0.5)\,(-1.5)(-25)}{3!}$$

$$+ \frac{(0.5)\,(-0.5)\,(-1.5)\,(-2.5)}{4!}\,(37) = 47.8672 \approx 48$$

f(45) - f(40) = 70 = Number of students who have scored between 41 and 45.

5) Find the interpolating polynomial for the following data:

f(0) = 1, f(1) = 0, f(2) = 1, f(3) = 10. Hence evaluate f(0.5)

x	y	Δ	Δ^2	Δ^3
0	1			
		-1		
1	0		2	
		1		6
2	1		8	
		9		
3	10			

$$u = \frac{x-0}{1} = x$$

$$f(x) = 1 + x(-1) + \frac{x(x-1)}{2!}(2) + \frac{x(x-1)\,(x-2)}{3!}\,6 = x^3 - 2x^2 + 1$$

6) Find the interpolating polynomial for the following data:

x:	0	1	2	3	4
f(x) :	3	6	11	18	27

x	y	Δ	Δ^2	Δ^3	Δ^4
0	3				
		3			
1	6		2		
		5		0	
2	11		2		0
		7		0	
3	18		2		
		9			
4	27				

$$u = \frac{x-0}{1} = x$$

$$f(x) = 3 + x(3) + \frac{x(x-1)}{2}(2) + \frac{x(x-1)}{x!}(0) \qquad = 3 + 2x + x^2$$

Newton Gregory Backward Interpolation formula

We use the following formula to calculate an approximate value of y= f(x) near the ending value of x_n of x as follows:

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + ...$$

$$\text{where} \quad u = \frac{x - x_n}{h}$$

1) The values of tan x are given for values of x in the following table. Estimate tan (0.26)

x	0.10	0.15	0.20	0.25	0.30
y	0.1003	0.1511	0.2027	0.2553	0.3093

x	y	∇	∇^2	∇^3	∇^4
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	0.3093				

$$u = \frac{0.26-0.3}{0.05} = -0.8$$

$$f(0.26) = 0.3093 + (-0.8)(0.054) + \frac{(-0.8)}{2}(0.2)\,(0.0014) + \frac{(-0.8)\,(0.2)\,(1.2)}{6}(0.0004) = 0.2659$$

2) The deflection d measured at various distances x from one end of a cantilever is given by the following table. Find d when x = 0.95

$$u = \frac{0.95-1}{0.2} = -0.25 \qquad d = 0.3308 \text{ when } x = 0.95$$

x	d	∇	∇^2	∇^3	∇^4	∇^5
0	0					
		0.0347				
0.2	0.0347		0.0479			
		0.0826		-0.0318		
0.4	0.1173		0.0161		0.0003	

		0.0987		-0.0321		-0.0003
0.6	0.2160		-0.016		0	
		0.0827		-0.032		
0.8	0.2987		-0.0481			
		0.0346				
1.0	0.3333					

3) The area y of circles for different diameters x are given below:

x :	80	85	90	95	100
y :	5026	5674	6362	7088	7854

Calculate area when x = 98

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
80	5026				
		648			
85	5674		40		
		688		-2	
90	6362		38		4
		726		2	
95	7088		40		
		766			
100	7854				

Answer:

$$u = \frac{x - x_n}{n} = -0.4$$

y = 7542

4) Find the interpolating polynomial which approximates the following data.

x	0	1	2	3	4

y	-5	-10	-9	4	35

x	y	∇	∇^2	∇^3	∇^4
0	-5				
		-5			
1	-10		6		
		1		6	
2	-9		12		0
		13		6	
3	4		18		
		31			
4	35				

$$u = \frac{x-4}{1}$$

$$f(x) = 35 + (x-4)\ (31) + (x-4)\ (x-3)\frac{18}{2!} \ + \frac{(x-4)\ (x-3)\ (x-2)\ (6)}{3!}$$

$$\boxed{f(x) \ = x^3 + 2x^2 + 6x - 5}$$

Interpolation with unequal intervals

Newton backward and forward interpolation is applicable only when x_0, x_1,…,x_{n-1} are equally spaced. Now we use two interpolation formulae for unequally spaced values of x.

i) **Lagranges formula for unequal intervals:**

If y = f(x) takes the values y_0, y_1, y_2,….,y_n corresponding to x = x_0, x_1, x_2,…,x_n then

$$f(x) = \frac{(x-x_1)\ (x-x_2)...(x-x_n)}{(x_0-x_1)\ (x_0-x_2)...(x_0-x_n)} f(x_0) \ + \frac{(x-x_0)\ (x-x_2)\ (x-x_3)...(x-x_n)}{(x_1-x_0)\ (x_1-x_2)\ (x_1-x_3)...(x_1-x_n)}\ f(x_1)$$

$$+ \frac{(x-x_0)\ (x-x_1)\ (x-x_3)...(x-x_n)}{(x_2-x_0)\ (x_2-x_1)\ (x_2-x_3)...(x_2-x_n)}\ f(x_2) + ...$$

$+\dfrac{(x-x_0)\,(x-x_1)\,(x-x_2)...(x-x_{n-1})}{(x_n-x_0)\,(x_n-x_1)\,(x_n-x_2)...(x_n-x_{n-1})}\,f(x_n)$ is known as the lagrange's interpolation formula

ii) Divided differences (Δ)

Let y_0,y_1……………………y_n be the values of the function y=f(x) corresponding values x_0,x_1……………………x_n which are not necessarily equally spaced.we define them as follows

$$\Delta f(x_0) = \Delta y_0 = \frac{y_1-y_0}{x_1-x_0} = [x_0,\, x_1]$$

$$\Delta y_1 = \frac{y_2-y_1}{x_2-x_1} = [x_2, x_1]$$

$$\Delta y_{n-1} = \frac{y_n-y_{n-1}}{x_n-x_{n-1}} = [x_{n-1}, x_n]$$

These are called as First divided difference.

The second divided difference are defined as follows:

$$\Delta^2 f(x_0) = \Delta^2 y_0 = \frac{\Delta y_1 - \Delta y_0}{x_2-x_0}$$

$$= \frac{[x_2,\, x_1]-[x_1,\, x_0]}{x_2-x_0} = [x_0,\, x_1,\, x_2]$$

$$\|^{ly}\ \Delta^2 y_1 = \frac{\Delta y_2-\Delta y_1}{x_3-x_1} = \frac{[x_3,x_2]-[x_2,\, x_1]}{x_3-x_1} = [x_1,\, x_2,\, x_3]$$

similarly $\Delta^3 y_0$,.... can be defined by following the above metod.

These divided differences may be employed to derive the following formula known as Newton's divided difference interpolation formula:

$$y = f(x) = y_0 + (x-x_0)\,\Delta y_0 + (x-x_0)\,(x-x_1)\,\Delta^2 y_0 + (x-x_0)\,(x-x_1)(x-x_2)\,\Delta^3 y_0$$

$$+ ... + (x-x_0)\,(x-x_1)\,...(x-x_n)\,\Delta^n y_0$$

.

Inverse interpolation: Finding the value of y given the value of x is called interpolation where as finding the value of x for a given y is called inverse interpolation.

Since Lagrange's formula is only a relation between x and y we can obtain the inverse interpolation formula just by interchanging x and y.

$$\therefore x = \frac{(y-y_1)(y-y_2)....(y-y_n)}{(y_0-y_1)(y_0-y_2)...(y_0=y_n)}.x_0$$

$$+\frac{(y-y_0)(y-y_0)(y-y_2)(y-y_3)...(y-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)...(y_1-y_n)}x_1+...$$

$$+...+\frac{(y-y_0)(y-y_1)...(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)...(y_n-y_{n-1})}.x_n$$

is the Lagranges formula for inverse interpolation

1) The following table gives the values of x and y

x :	1.2	2.1	2.8	4.1	4.9	6.2
y :	4.2	6.8	9.8	13.4	15.5	19.6

Find x when y = 12 using Lagranges inverse interpolation formula.

Using Langrages formula

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)(y-y_5)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)(y_0-y_4)(y_0-y_5)}x_0$$

$$+....+\frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_5-y_0)(y_5-y_1)(y_5-y_2)...(y_5-y_4)}x_4$$

$$=0.022-0.234+1.252+3.419-0.964+0.055$$

$$= 3.55$$

2) Given the values

x :	5	7	11	13	17
f(x) :	150	392	1452	2366	5202

Evaluate f(9) using (i) Lagrange's formula (ii) Newton's divided difference formula.

i) Lagranges formula

$$f(9) = \frac{(9-7)\,(9-11)\,(9-13)\,(9-17)}{(5-7)\,(5-11)\,(5-13)\,(5-17)}\,(150) + \frac{(9-5)\,(9-11)\,(9-13)\,(9-17)}{(7-5)\,(7-11)\,(7-13)\,(7-17)}\,.392$$

$$+\frac{(9-5)\,(9-7)\,(9-13)\,(9-17)}{(11-5)\,(11-7)\,(11-13)\,(11-17)}(1452) + \frac{(9-5)\,(9-7)\,(9-11)\,(9-17)}{(13-5)\,(13-7)\,(13-11)\,(13-17)}\,(2366)$$

$$+\frac{(9-5)\,(9-7)\,(9-11)\,(9-13)}{(17-5)\,(17-7)\,(17-11)\,(17-13)}(5202) = 810$$

f(9) = 810

ii)

5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0
		457		1	
13	2366		42		
		709			
17	5202				

f(9) = 150 + 121 (9 - 5) + 24 (9 - 5) (9 - 7) + 1(9 - 5) (9 - 7) (9 - 11) = 810

3) Using i) Langranges interpolation and ii) divided difference formula. Find the value of y when x = 10.

x :	5	6	9	11
y :	12	13	14	16

i) Lagranges formula

$$y = f(10) = \frac{(10-6)\ (10-9)\ (10-11)}{(5-6)\ (5-9)\ (5-11)} \times 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16$$

$$= \frac{44}{3}$$

ii)Divided difference

X	y	Δ	Δ^2	Δ^3
5	12			
		1		
6	13		$-\frac{2/3}{4} = \frac{-1}{6}$	
		$\frac{1}{3}$		$\frac{\frac{2}{15}+\frac{1}{6}}{11-5} = \frac{\frac{27}{90}}{6} = \frac{3/10}{6} = \frac{1}{20}$
9	14		$\frac{2/3}{5} = \frac{2}{15}$	
		$\frac{2}{2} = 1$		
11	16			

$$f(10) = 12 + (10-5) + (10-5)(10-6).\left(-\frac{1}{6}\right) + (10-5)(10-6)(10-9)\left(\frac{1}{20}\right)$$

$$= \frac{44}{3}$$

4) If y(1) = -3, y(3) = 9, y(4) =30, y(6) = 132 find the lagranges interpolating polynomial that takes the same values as y at the given points.

Given:

X	1	3	4	6
Y	-3	9	30	132

$$f(x) = \frac{(x-3)\,(x-4)\,(x-6)}{(1-3)(1-4)(1-6)}\,.\,(-3) + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)}.9$$

$$+ \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)}.30 + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)}.132$$

$$= x^3 - 3x^2 + 5x - 6$$

5) Find the interpolating polynomial using Newton divided difference formula for the following data:

X	0	1	2	5
Y	2	3	12	147

X	y	Δ	Δ^2	Δ^3
0	2			
		1		
1	3		4	
		9		1

2	12		9	
		45		
5	147			

F(x) = 2 + (x - 0)(1) + (x - 0) (x - 1) (4) + (x - 0) (x - 1) (x - 2) 1

$= x^3 + x^2 - x + 2$

Numerical Integration:-

Evaluating the value of $I = \int_a^b y\,dx$ numerically, given the set of values (x_i, y_i), $i = 0,1,2,...,n$ at regular intervals is known as Numerical Integration.The following formulae can be used to Evaluate the integral numerically.

(i) Simpson's one third rule:-

$$I = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2})\right]$$

when n is even.

(ii) Simpson's three-eighth rule:-

$$I = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + ... + y_{n-1}) + 2(y_3 + y_6 + ... + y_{n-3})\right]$$

when n is a multiple of 3.

(iii) Weddle's rule:-

$$I = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6 + ...\right]$$

when n is a multiple of 6.

Problems:

1) Using Simpson's $\frac{1}{3}^{rd}$ rule evaluate $\int_0^1 \frac{dx}{1+x^2}$ by dividing the interval $(0, 1)$ into 4 equal sub intervals and hence find the value of π correct to four decimal places.

Solution: Let us divide [0,1] into 4 equal strips (n = 4)

$\therefore$ length of each strip: $h = \frac{1-0}{4} = \frac{1}{4}$

The points of division are $x = 0, \frac{1}{4}, \frac{2}{4} = \frac{1}{2}, \frac{3}{4}, \frac{4}{4} = 1$

By data $y = \frac{1}{1+x^2}$

Now we have the following table

x	0	1/4	½	3/4	1
$y = \frac{1}{1+x^2}$	1	16/17	4/5	16/25	½
	y_0	y_1	y_2	y_3	y_4

Simpson's $\frac{1}{3}^{rd}$ rule for n = 4 is given by

$$\int_a^b y\,dx = \frac{h}{3}\left[(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)\right]$$

$$\therefore \int_0^1 \frac{1}{1+x^2}\,dx = \frac{1/4}{3}\left[\left(1+\frac{1}{2}\right) + 4\left(\frac{16}{17} + \frac{16}{25}\right) + 2.\frac{4}{5}\right] = 0.7854$$

$$Thus \quad \int_0^1 \frac{1}{1+x^2}\,dx = 0.7854$$

To deduce the value of π: We perform theoretical integration and equate the resulting value to the numerical value obtained.

$$\therefore \int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}$$

$$We\,must\,have,\ \frac{\pi}{4} = 0.7854 \Rightarrow \pi = 4(0.7854) = 3.1416$$

$$Thus \quad \boxed{\pi = 3.1416}$$

2) Given that

x	4	4.2	4.4	4.6	4.8	5	5.2
$\log x$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

Evaluate $\int_4^{5.2} \log x \, dx$ using Simpson's $\frac{3}{8}^{th}$ rule

Solution: Simpson's $\frac{3}{8}^{th}$ rule for n = 6 is given by

$$\int_a^b ydx = \frac{3h}{8}\left[\left(y_0+y_6\right)+3\left(y_1+y_2+y_4+y_5\right)+2\left(y_3\right)\right]$$

$$\int_4^{5.2} \log_e xdx = \frac{3(0.2)}{8}\left[\left(1.3863+1.6487\right)+3\left(1.4351+1.4816+1.5686+1.6094\right)+2\left(1.5261\right)\right]$$

$$\int_4^{5.2} \log_e xdx = 1.8279$$

3) Using Weddle's rule evaluate $\int_0^1 \frac{xdx}{1+x^2}$ by taking seven ordinates and hence find $\log_e 2$

Solution: Let us divide [0,1] into 6 equal strips (since seven ordinates)

$\therefore$ length of each strip: $h = \frac{1-0}{6} = \frac{1}{6}$

The points of division are $x = 0, \frac{1}{6}, \frac{2}{6} = \frac{1}{3}, \frac{3}{6} = \frac{1}{2}, \frac{4}{6} = \frac{2}{3}, \frac{5}{6}, \frac{6}{6} = 1$

By data $y = \frac{1}{1+x^2}$

Now we have the following table

x	0	1/6	1/3	½	2/3	5/6	1
$y = \frac{x}{1+x^2}$	0	6/37	3/10	2/5	6/13	30/61	½
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Weddle's rule for n = 6 is given by

$$\int_a^b ydx = \frac{3h}{10}\left[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6\right]$$

$$\int_0^1 \frac{x}{1+x^2}dx = \frac{3(1/6)}{10}\left[0+5(6/37)+3/10+6(2/5)+6/13+5(30/61)+1/2\right]$$

$$\int_0^1 \frac{x}{1+x^2}dx = 0.3466$$

To deduce the value of $\log_e 2$: We perform theoretical integration and equate the resulting value to the numerical value obtained.

$$\therefore \int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\log_e(1+x^2)\Big]_0^1 = \frac{1}{2}\log_e 2 - \frac{1}{2}\log_e 1$$

Hence $\displaystyle\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\log_e 2$

We must have, $\displaystyle\frac{1}{2}\log_e 2 = 0.3466 \Rightarrow \log_e 2 = 2(0.3466) = 0.6932$

Thus $\boxed{\log_e 2 = 0.6932}$

Solution of Algebraic and Transcendental Equations

The equation f(x) = 0 , is called as Transcedental equation , if it contains algebraic function or trigonometric function or both.

Ex: (1) $x^4 - 7x^3 + 3x + 5 = 0$ is transcedental

(2) $e^x - x \tan x = 0$ is transcendental

The approximate root for an Transcedental equation is fond by the following two iterative methods:

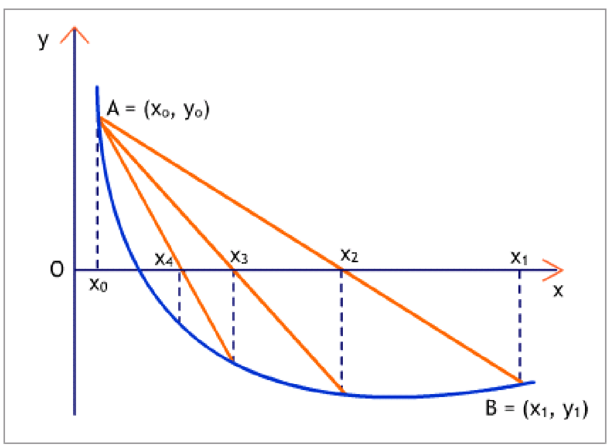
I. Regula-Falsi Method.

II. Newton Raphson's Method

Method of false position or Regula-Falsi Method:

This is a method of finding a real root of an equation f(x) = 0 and is slightly an improvisation of the bisection method.

Let x_0 and x_1 be two points such that $f(x_0)$ and $f(x_1)$ are opposite in sign.



Let $f(x_0) > 0$ and $f(x_1) < 0$

The graph of y = f(x) crosses the x-axis between x_0 and x_1

$\therefore$ Root of f(x) = 0 lies between x_0 and x_1

Now equation of the Chord AB is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \quad ...(1)$$

When y =0 we get x = x_2

$$\text{i.e. } x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \quad ...(2)$$

Which is the first approximation

If $f(x_0)$ and $f(x_2)$ are opposite in sign then second approximation

$$x_3 = x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_0)$$

This procedure is continued till the root is found with desired accuracy.

Poblems:

1. Find a real root of x^3 - 2x -5 = 0 by method of false position correct to three decimal places between 2 and 3.

Answer:

Let $f(x) = x^3 - 2x - 5 = 0$

$f(2) = -1$

$f(3) = 16$

$\therefore$ a root lies between 2 and 3

Take $x_0 = 2$, $x_1 = 3$

$\therefore$ $x_0 = 2$, $x_1 = 3$

$$\text{Now } x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2 - \frac{3 - 2}{16 + 1}(-1)$$

$= 2.0588$

$f(x_2) = f(2.0588) = -0.3908$

$\therefore$ Root lies between 2.0588 and 3

Taking $x_0 = 2.0588$ and $x_1 = 3$

$f(x_0) = -0.3908$, $f(x_1) = 16$

$$\text{We get } x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} .f(x_0)$$

$$= 2.0588 - \frac{0.9412}{16.3908}(-0.3908)$$

$= 2.0813$

$f(x_3) = f(2.0813) = -0.14680$

$\therefore$ Root lies between 2.0813 and 3

Taking $x_0 = 2.0813$ and $x_1 = 3$

$f(x_0) = 0.14680$, $f(x_1) = 16$

$$x_4 = 2.0813 - \frac{0.9187}{16.1468}(-0.14680) = 2.0897$$

Repeating the process the successive approximations are

$x_5 = 2.0915$, $x_6 = 2.0934$, $x_7 = 2.0941$, $x_8 = 2.0943$

Hence the root is 2.094 correct to 3 decimal places.

2. Find the root of the equation $xe^x = \cos x$ using Regula falsi method correct to three decimal places.

Solution:

Let $f(x) = \cos x - xe^x$

Observe

$f(0) = 1$

$f(1) = \cos 1 - e = -2.17798$

$\therefore$ root lies between 0 and 1

Taking $x_0 = 0$, $x_1 = 1$

$f(x_0) = 1$, $f(x_1) = -2.17798$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)}.f(x_0)$$

$$= 0 - \frac{1}{-3.17798}(1) = 0.31467$$

$f(x_2) = f(0.31467) = 0.51987$ +ve

$\therefore$ Root lies between 0.31467 and 1

$x_0 = 0.31467$, $x_1 = 1$

$f(x_0) = 0.51987$, $f(x_1) = -2.17798$

$$x_3 = 0.31467 - \frac{1 - 0.31467}{-2.17798 - 0.51987}(0.51987) = 0.44673$$

$f(x_3) = f(0.44673) = 0.20356$ +ve

$\therefore$ Root lies between 0.44673 and 1

$$x_4 = 0.44673 + \frac{0.55327}{2.38154} \times 0.20356 = 0.49402$$

Repeating this process

$x_5 = 0.50995$, $x_6 = 0.51520$, $x_7 = 0.51692$, $x_8 = 0.51748$
$x_9 = 0.51767$, etc

Hence the root is 0.518 correct to 4 decimal places

Newton Raphson Method

This method is used to find the isolated roots of an equation f(x) = 0, when the derivative of f(x) is a simple expression.

Let m be a root of f(x) = 0 near a.

$\therefore \quad f(m) = 0$

We have by Taylor's series

$$f(x) = f(a) + (x-a)\, f'(a) + \frac{(x-a)^2}{2!}\, f''(a) + \ldots..$$

$$\therefore \quad f(m) = f(a) + (m-a)\, f'(a) + \ldots..$$

Ignoring higher order terms

$$f(m) = f(a) + (m - a)\, f'(a) = 0$$

$$\text{or } m - a = -\frac{f(a)}{f'(a)}$$

$$\text{or } m = a - \frac{f(a)}{f'(a)}$$

Let $a = x_0$, $m = x_1$

then $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ is the first approximation

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ is the second approximation

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$x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$ is the iterative formula for Newton Raphson Method

1. Using Newton's Raphson Method find the real root of $x \log_{10} x = 1.2$ correct to four decimal places.

Answer:

Let $f(x) = x \log_{10} x - 1.2$

$f(1) = -1.2$, $f(2) = -0.59794$, $f(3) = 0.23136$

$$\text{We have } f(x) = \frac{x \log_e x}{\log_e 10} - 1.2 \Rightarrow f'(x) = \frac{1 + \log_e x}{\log_e 10}$$

$$= \log_{10} e + \log_{10} x$$

$$\therefore \quad x_{k+1} = x_k - \frac{x_k \log_{10} x_k - 1.2}{\log_{10} e + \log_{10} x_K}$$

Let $x_0 = 2.5$ (you may choose 2 or 3 also)

$$x_1 = 2.5 - \frac{2.5 \log_{10} 2.5 - 1.2}{\log_{10} e + \log_{10} 2.5} = 2.7465$$

$$x_2 = 2.7465 - \frac{2.7465 \log 2.7465 - 1.2}{\log_{10} e + \log_{10} 2.7465} = 2.7406$$

Repeating the procedure

$$x_3 = 2.7406$$

$$\therefore \quad x \approx 2.7406 \text{ is the root of the given equation}$$

2.Using Newton's Method, find the real root of $xe^x = 2$. Correct to 3 decimal places.

Answer:

Let $f(x) = xe^x - 2$

$f(0) = -2$

$f(1) = e - 2 = 0.7182$

Let $x_0 = 1$

$f'(x) = (x + 1) e^x$

We have

$$x_{k+1} = x_k - \frac{x_k e^{x_k} - 2}{(x_k + 1)\, e^{x_k}}$$

$$x_1 = 1 - \frac{e - 2}{2e} = 0.8678$$

$$x_2 = 0.8678 - \frac{(0.8678)\, e^{0.8678} - 2}{(1.8678)\, e^{0.8678}} = 0.8527$$

$$x_3 = 0.8527 - \frac{(0.8527)\, e^{0.8527} - 2}{(1.8527)\, e^{0.8527}} = 0.8526$$

$\therefore \quad x \approx 0.8526$, is the required root. Correct to 3 decimal places

3. Find by Newton's Method the real root of 3x = cosx + 1 near 0.6, x is in radians. Correct for four decimal places.

Answer:

Let f(x) = 3x - cosx - 1

$f'(x) = 3 + \sin x$

$$x_{k+1} = x_k - \frac{3x_k - \cos x_{k-1}}{3 + \sin x_k}$$

When $x_0 = 0.6 \quad x_1 = 0.6 - \dfrac{3(0.6) - \cos(0.6) - 1}{3 + \sin(0.6)} = 0.6071$

$$x_2 = 0.6071 - \frac{3(0.6071) - \cos(0.6071) - 1}{3 + \sin(0.6071)} = 0.6071$$

Since $x_1 = x_2$

The desired root is 0.6071

4. Obtain the iterative formula for finding the square root of N and find $\sqrt{41}$

Answer:

Let $x = \sqrt{N}$

or $x^2 - N = 0$

$\therefore f(x) = x^2 - N$

$f'(x) = 2x$

Now

$$x_{k+1} = x_k - \frac{x_k^2 - N}{2x_k}$$

$$= x_k - \frac{x_k}{2} + \frac{N}{2x_k}$$

i.e. $x_{k+1} = \dfrac{1}{2}\left\{x_k + \dfrac{N}{x_k}\right\}$

To find $\sqrt{41}$

Observe that $\sqrt{36} < \sqrt{41}$

$\therefore$ Choose $x_0 = 6$

$$x_1 = \frac{1}{2}\left\{6 + \frac{41}{6}\right\} = 6.4166$$

$$x_2 = \frac{1}{2}\left\{6.4166 + \frac{41}{6.4166}\right\} = 6.4031$$

$$x_3 = \frac{1}{2}\left\{6.4031 + \frac{41}{6.4031}\right\} = 6.4031$$

Since $x_2 = x_3 = 6.4031$

The value of $\sqrt{41} \approx 6.4031$

5. Obtain an iterative formula for finding the p-th root of N and hence find $(10)^{1/3}$ correct to 3 decimal places.

Answer:

Let $x^p = N$

or $x^p - N = 0$

Let $f(x) = x^p - N$

$f'(x) = px^{p-1}$

Now $x_{k+1} = x_k - \dfrac{x_k^p - N}{px_k^{p-1}}$

Observe that $8 < 10$

$\Rightarrow \quad 8^{1/3} < 10^{1/3}$

i.e. $\quad 2 < (10)^{1/3}$

$\therefore$ Use $x_0 = 2$, $p = 3$, $N=10$

$$x_1 = 2 - \frac{2^3 - 10}{3(2^2)} = 2.1666$$

$$x_2 = 2.1666 - \frac{(2.1666)^3 - 10}{3(2.1666)^2} = 2.1545$$

$$x_3 = 2.1545 - \frac{(2.1545)^3 - 10}{3(2.1545)^2} = 2.1544$$

$\therefore \quad (10)^{1/3} \approx 2.1544$

6. Obtain an iterative formula for finding the reciprocal of p-th root of N. Find $(30)^{-1/5}$ correct to 3 decimal places.

Answer:

Let $x^{-p} = N$

or $x^{-p} - N = 0$

$\therefore$ $f(x) = x^{-p} - N$

$f'(x) = -px^{-p-1}$

Now

$$x_{k+1} = x_k + \frac{x_k^{-p} - N}{p\, x_k^{-p-1}}$$

$$\text{since } (32)^{-1/5} = \frac{1}{2} = 0.5$$

We use $x_0 = 0.5$, $p = 5$, $N = 30$

$$x_1 = 0.5 + \frac{(0.5)^{-5} - 30}{5(0.5)^{-6}} = 0.50625, \text{ Repeating the process}$$

$$x_2 = 0.506495, \; x_3 = 0.506495$$

$$\therefore \qquad (30)^{-1/5} \approx 0.5065$$