06IP/IM74 OPERATIONS RESEARCH

UNIT - 5: Queuing Theory

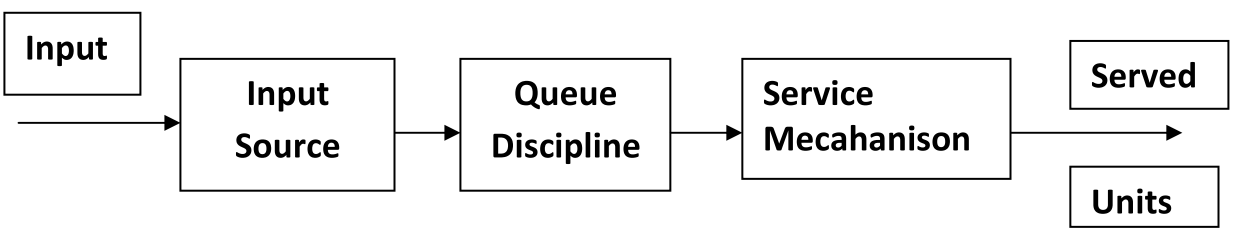
(By **Dr.NVR Naidu**, Prof & HOD (IEM), MSRIT, Bangalore)

The Basic structure of queuing model

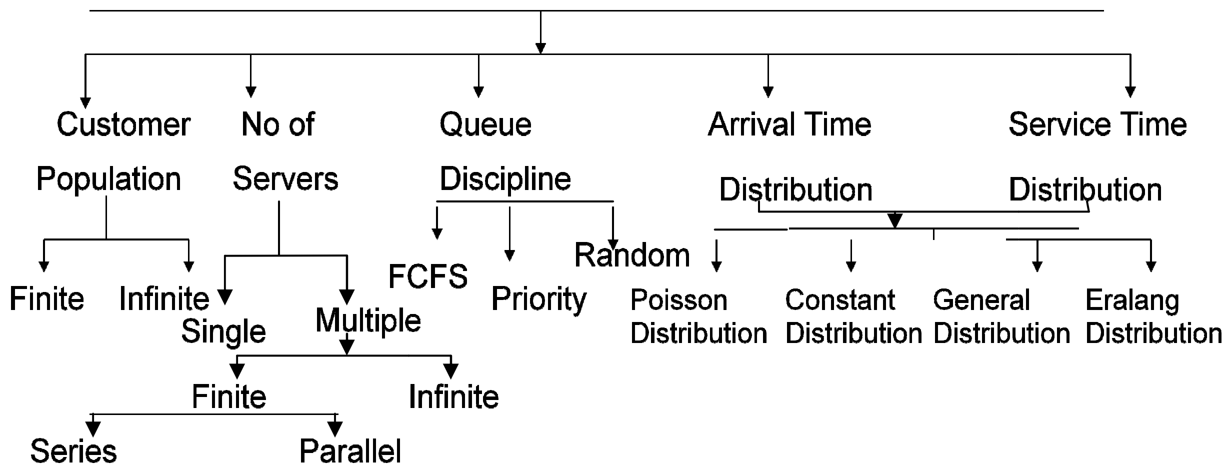
Introduction

Queues are a part of everyday life. We all wait in queues to buy a movie ticket, to make bank deposit, pay for groceries, mail a package, obtain a food in a cafeteria, to have ride in an amusement park and have become adjustment to wait but still get annoyed by unusually long waits.

The Queuing models are very helpful for determining how to operate a queuing system in the most effective way if too much service capacity to operate the system involves excessive costs. The models enable finding an appropriate balance between the cost of service and the amount of waiting.



Information required to solve the queuing problem:



Characteristics of the queuing system-:

(a) Input source

(b) Queue discipline

(c) Service mechanism

(a) Input source

One characteristic of the input source is the size. The size is the total number of units that might require service from time to time. It may be assumed to be finite or infinite.

The customer assumption is that they generate according to 'Poisson Distribution' at a certain average rate

Therefore, the equivalent assumption is that they generate according to exponential distribution between consecutive arrivals. To solve the problems use & assume customer population as ∞

(b) Queue Discipline

A queue is characterized by maximum permissible number of units that it contains. Queues are called finite or infinite, according to whether number is finite or infinite. The service discipline refers to the order in which number of queues are selected for service.

Ex: It may be FIFO, random or priority; FIFO is usually assumed unless stated otherwise.

(c) Service mechanism

This consists of one or more service facilities each of which contains one or more parallel service channel. If there is more than one service facility, the arrival unit may receive the service from a sequence of service channels.

At a given facility, the arrival enters at the service facility and is completely served by that server. The time elapsed from the commencement of the service to its completion for an unit at the service facility is known as service time usually, service time follows as exponential distribution.

Classification of queuing models using kendal & Lee notations

Generally, any queuing models may be completely specified in the following symbolic form

$$a / b / c : d / e$$

a $\longrightarrow$ Type of distribution of inter – arrival time

b $\longrightarrow$ Type of distribution of inter – service time

c $\longrightarrow$ Number of servers

d $\longrightarrow$ Capacity of the system

e $\longrightarrow$ Queue discipline

M $\longrightarrow$ Arrival time follows Poisson distribution and service time follows an exponential distribution.

Model I : M / M / 1 : / FCFS

Where M Arrival time follows a Poisson distribution

M $\rightarrow$ Service time follows a exponential distribution

1 $\rightarrow$ Single service model

∞ $\rightarrow$ Capacity of the system is infinite

FCFS $\rightarrow$ Queue discipline is first come first served

Model II : M / M / 1 : N / FCFS

Where N $\rightarrow$ Capacity of the system is finite

Model III : M / M / 1 : / SIRO

Where SIRO $\rightarrow$ Service in random order

Model IV : M / O / 1 : / FCFS

Where D $\rightarrow$ Service time follows a constant distribution or is deterministic

Model V : M / G / 1 : / FCFS

Where G $\rightarrow$ Service time follows a general distribution or arbitrary distribution

Model VI : M / E_k / 1 : / FCFS

Where E_k $\rightarrow$ Service time follows Erlang distribution with K phases.

Model VII : M / M / K : / FCFS

Where K $\rightarrow$ Multiple Server model

Model VIII : M / M / K : N / FCFS

Model I: M / M /1 : / FCFS

Formulas: 1. Utilization factor traffic intensity /

Utilization parameter / Busy period

$$\rho = \frac{\lambda}{\mu}$$

Where λ = Mean arrival rate ; μ= mean service rate

Note : $\mu > \lambda$ in single server model only

2. Probability that exactly zero units are in the system

$$P_o = 1 - \frac{\lambda}{\mu}$$

3. Probability that exactly 'n' units in the system

$$P_n = P_o \left(\frac{\lambda}{\mu} \right)^n$$

4. Probability that n or more units in the system

$$P_{n\ or\ more} = \left(\frac{\lambda}{\mu} \right)^n$$

more then 'n' means n should be n+1

5. Expected number of units in the queue / queue length

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

6. Expected waiting time in the queue

$$W_q = \frac{L_q}{\lambda}$$

7. Expected number of units in the system

$$L = L_q + \frac{\lambda}{\mu}$$

8. Expected waiting time in the system

$$W = W_q + \frac{1}{\mu}$$

9. Expected number of units in queue that from time to time – (OR) non - empty queue size

$$D = \frac{\mu}{\mu - \lambda}$$

10. Probability that an arrival will have to wait in the queue for service

Probability $= 1 - P_o$

11. Probability that an arrival will have to wait in the queue more than w (where w > o), the waiting time in the queue

Probability $= \left(\frac{\lambda}{\mu}\right) e^{(\lambda-\mu)w}$

12. Probability that an arrival will have to wait more than v(v > o) waiting time in the system is

$$= e^{(\lambda-\mu)^{v}}$$

13. Probability that an arrival will not have to wait in the queue for service = P_o

Model 1 - Problems

1. Arrivals at a telephone both are considered to be Poisson at an average time of 8 min between our arrival and the next. The length of the phone call is distributed exponentially, with a mean of 4 min.

Determine

(a) Expected fraction of the day that the phone will be in use.

(b) Expected number of units in the queue Expected waiting time in the queue.

(c) Expected number of units in the system.

(e) Expected waiting time in the system

(f) Expected number of units in queue that from time to time.

(g) What is the probability that an arrival will have to wait in queue for service?

(h) What is the probability that exactly 3 units are in system

(i) What is the probability that an arrival will not have to wait in queue for service?

(j) What is the probability that there are 3 or more units in the system?

(k) What is the probability that an arrival will have to wait more than 6 min in queue for service?

(l) What is the probability that more than 5 units in system

(m) What is the probability that an arrival will have to wait more than 8 min in system?

(n) Telephone company will install a second booth when convinced that an arrival would have to wait for attest 6 min in queue for phone. By how much the flow of arrival is increased in order to justify a second booth.

Solution:

The mean arrival rate = λ = 1/8 x 60 = 7.5 / hour.

The mean service = μ = x 60 = 15 / hour.

a) Fraction of the day that the phone will be in use

$$\rho = \frac{\lambda}{\mu} = \frac{7.5}{15} = 0.5$$

(b) The expected number pf units in the queue

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{7.5^2}{15(15-7.5)}$$

$$L_q = 0.5\,(units)\,person$$

(c) Expected waiting time in the queue

$$W_q = \frac{L_q}{\lambda}$$

$$= \frac{0.5}{7.5} = 0.066\ hrs$$

(d) Expected number of units in the system:-

$L = L_q + \lambda / \mu$

$= 0.5+0.5$

$L = 1$ person

(e) Expected waiting time in the system

$$W = Wq + \frac{1}{\mu}$$

$$= 0.066 + \frac{1}{15} = 0.133$$

(f) Expected number of units in the queue that form from time to time:-

$$D = \frac{\mu}{\mu - \lambda}$$

$$= \frac{15}{15-7.5} = 2\ \ persons$$

(g) Probability that an arrival will have to wait in the system:-

$$P_{ro} = 1 - P_o$$

$$P_o = 1 - \frac{\lambda}{\mu}$$

$$= 1 - \left(1 - \frac{\lambda}{\mu}\right)$$

$$P_{ro} = \frac{\lambda}{\mu} = \frac{7.5}{15} = 0.5$$

(h) The Probability that exactly zero waits in the system:-

$$P_o = 1 - \frac{\lambda}{\mu}$$

$$= 1 - 0.5 = 0.5$$

(i) The probability that exactly 3 units in the system:-

$$P_n = P_o - \left(\frac{\lambda}{\mu}\right)^n \qquad n = 3$$

$$P_3 = 0.5(0.5)^3 = 0.0625$$

(j) Probability that an arrival will not have to wait for service:-

$$P_o = 1 - \frac{\lambda}{\mu}$$

$$= 0.5$$

(k) Probability that 3 or more units in the system:-

$$P_{n\ or\ more} = \left(\frac{\lambda}{\mu}\right)^n \qquad n = 3$$

$$P_{n\ or\ more} = 0.5^3 = 0.125$$

(l) Probability that an arrival will have to wait more than 6mins in queue for service

$$P_{ro} = \left(\frac{\lambda}{\mu}\right) e^{(\lambda-\mu)\omega}$$

$$\omega = 6\min = \frac{6}{60} hrs$$

$$P_{ro} = 0.5\, e^{(7.5-15)\frac{6}{60}}$$

$$P_{ro} = 0.236$$

(m) Probability that more than 5 units in the system

$$P_{ro} = \left(\frac{\lambda}{\mu}\right)^n \qquad n = 6$$

$$P_{ro} = 0.5^6 = 0.015$$

(n) Probability that an arrival will directly enter for service

P_o= 0.5

(O) Probability that arrival will have to wait more than 8mins in the system.

V = 8 / 60 hrs

$$P_{ro} = e^{(\lambda-\mu)v}$$

$$= e^{(7.5-15)\frac{8}{60}}$$

$$= 0.367$$

(p)

$$W_q = \frac{6}{60}\,hrs \quad = 0.1\,hr$$

$$\therefore \quad W_q = \frac{L_q}{\lambda} = \frac{\lambda^2}{\mu(\mu-\lambda)\lambda}$$

$$0.1 = \frac{\lambda}{15(15-\lambda)}$$

$$\therefore \quad \lambda = 9\,per\ hour.$$

To justify a second booth should be increased from 7.5 to 9 per hour

2) In a self service store with one cashier, 8 customers arrive on an average of every 5 mins. and the cashier can serve 10 in 5 mins. If both arrival and service time are exponentially distributed, then determine

a) Average number of customer waiting in the queue for average. (3.2)

b) Expected waiting time in the queue (0.033)

c) What is the probability of having more than 6 customers In the system (0.209)

Solution:

Mean arrival rate =λ= =1.6 x 60

= 96 / hour

Mean service rate = μ = x 60

= 120 / hour.

(a) Average number of customers waiting in queue for service

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{96^2}{120(120-96)}$$

$$L_q = 3.2 \; customers$$

(b) Expected waiting time in the queue

$$W_q = \frac{L_q}{\lambda} = \frac{3.2}{96} = 0.033$$

(c) Probability of having more than 6 customers in the system

$$P_{6\, or\, more} = \left(\frac{\lambda}{\mu}\right)^n \qquad where \quad n = 7$$

$$= \left(\frac{96}{120}\right)^7 = 0.209$$

3) Consider a box office ticket window being manned by a single server. Customer arrives to purchase ticket according to Poisson input process with a mean rate of 30/hr. the time required to serve a customer has an ED with a mean of 90 seconds determine:

(a) Mean queue length. (2.25)
(b) Mean waiting time in the system. (0.1)
(c) The probability of the customer waiting in the queue for more than 10min. (0.1416)
(d) The fraction of the time for which the server is busy. (0.75)

Solution:

The mean arrival rate = $\lambda = \quad 30/hr$

The mean service rate

$$\mu = \frac{1}{90} \times 60 \times 60$$
$$= 40/hr$$

(a) Mean queue length

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} \quad \frac{30^2}{40(40-30)} \quad = 2.25\ customers$$

(b) Mean waiting time in the system

$$W = W_q + \frac{1}{\mu}$$
$$= \frac{L_q}{\lambda} + \frac{1}{\mu}$$
$$= \frac{2.25}{30} + \frac{1}{40}$$
$$= 0.1hr$$

(c) Probability of the customer waiting in queue for more than 10min.

$$W\frac{10}{60} \quad = 1/6hour$$

$$P_{ro} = \left(\frac{\lambda}{\mu}\right)^{e^{(\lambda-\mu)w}}$$

$$= \left(\frac{30}{40}\right)^{e^{(30-40)1/6}}$$

$$P_{ro} = 0.1416$$

(d) Fraction of time the serve is busy

$$\rho = \frac{\lambda}{\mu}$$

$$= \frac{40}{30}$$

$$= 0.75hr$$

4) A T.V repairman repair the sets in the order in which they arrive and expects that the time required to repair a set has an ED with mean 30mins. The sets arrive in a Poisson fashion at an average rate of 10/8 hrs a day.

(a) What is the expected idle time / day for the repairman? (0.375x8)

b) How many TV sets will be there awaiting for the repair? (1.04)

Solution

Mean arrival rate = λ = $\frac{10}{8}$ hours

Mean service rate = μ = $\frac{1}{30}$ x 60 = 2 hours

(a) Expected idle time / day of the repair

Busy Period = $\frac{\lambda}{\mu}$ = $\frac{1.25}{2}$ = 0.625 hour

$$\therefore idle\ time = P_o = 1 - \frac{\lambda}{\mu} = 1 - 0.625 = 0.375$$

$$\therefore idle\ time / day = 0.375 \times 8 = 3\ hrs / day$$

(b) Number of T.V sets awaiting for the repair:-

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{1.25^2}{2(2 - 1.25)} = 1.04$$

5) In a bank there is only on window. A solitary employee performs all the service required and the window remains continuously open from 7am to 1pm. It has discovered that an average number of clients is 54 during the day and the average service time is 5mins / person. Find

a) Average number of clients in the system (3)
b) Average waiting time (0.25)
c) The probability that a client has to spend more than 10mins in a system. (0.60)

Solution

The mean arrival rate = $$\lambda = \frac{54}{6}$$
$$= 9 \; clients/hour$$

The mean service rate $$= \frac{1}{5} \times 60$$
$$= 12 \; clients/hour$$

(a) Average number of customer in the system

$$L = \quad L_q + \frac{\lambda}{\mu}$$
$$= \frac{\lambda}{\mu(\mu - \lambda)} + \frac{\lambda}{\mu}$$
$$= \frac{9^2}{12\,(12-9)} + \frac{9}{12}$$
$$L = 3 \; clients$$

(b) Average waiting time:-

$$W_q = \frac{L_q}{\lambda} \quad = \frac{\lambda^2}{\mu(\mu-\lambda)\lambda} \quad = \frac{9}{12(12-9)} \quad = 0.25$$

(c) Probability that a customer has to spend more than 10min in a system.

$$\vartheta = \frac{10}{60} \qquad = 1/6hr$$
$$P_{ro} = e^{(\lambda-\mu)v} \qquad = e^{(9-12)1/6} \qquad = 0.606$$

6) A departmental Secretary receive an average of 8 job / hr. many are short jobs, while other are quiet long. Assume however, that the time to perform a job has an ED mean of 6mins determine

a) The average elapsed time from the time the secretary receives a job, until it is completed. (0.5)

b) Average number of jobs in a system (4)

c) The probability that the time in the system is greater than ½hr. (0.3621)

d) Probability of more than 5 jobs in the system. (0.2628)

Solution

Mean arrival rate = λ = 8 jobs / hrs

Mean service rate = μ = x 60

= 10 jobs / hrs.

(a) Average elapsed time from the time the secretary receives a job on till it is completed

$$W = Wq + \frac{1}{\mu}$$

$$= \frac{Lq}{\lambda} + \frac{1}{\mu}$$

$$= \frac{\lambda^2}{\mu(\mu-\lambda)\lambda} + \frac{1}{\mu}$$

$$= \frac{8}{10(10-8)} + \frac{1}{10} = 0.5$$

(b) Average number of jobs in the system:-

$$L = L_q + \frac{\lambda}{\mu}$$

$$= \frac{\lambda^2}{\mu(\mu-\lambda)} + \frac{\lambda}{\mu}$$

$$= \frac{8^2}{10(10-8)} + \frac{8}{10}$$

$$L = 4 \ jobs$$

(c) Probability that the customer spends time in the system is greater than ½ hr.

$$v = 0.5 \ hr$$

$$P_{ro} = e^{(\lambda-\mu)v}$$

$$= e^{(8-10)0.5}$$

$$= 0.367$$

(d) Probability of more than 5 jobs in the system:-

$$P_{ro} = \left(\frac{\lambda}{\mu}\right)^n \qquad n = 6$$

$$P_{ro} = \frac{8}{10}^6 = 0.262$$

7) At public telephone booth in a post – office arrivals are considered to be Poisson fashion with an average inter arrival time of 12mins. The length of the phone call is ED with a mean of 4mins. Determine:

(a) The probability that the fresh arrival will not have to wait for the phone. (0.66)

(b) What is the probability that the an arrival will have to wait for more than 10mins before the phone is free (0.0629)

(c) What is the average length of the queue that forms from time to time (1.5)

Solution:

Mean arrival rate = λ = $\frac{1}{12}$ x 60 = 5 / hr

Mean service rate = μ = $\frac{1}{4}$ x 60 = 15 / hr

(a) Probability that fresh arrivals will not have to wait for the phone:

$$
\begin{aligned}
P_{ro} &= P_o \\
&= 1 - \frac{\lambda}{\mu} \\
&= 1 - \frac{5}{15} \\
&= 0.66
\end{aligned}
$$

(b) Probability that an arrival will have to wait more than 10 min before the phone is free :

$$
\begin{aligned}
W &= \frac{10}{60} = \frac{1}{6} hr \\
P_{ro} &= \left(\frac{\lambda}{\mu}\right) e^{(\lambda-\mu)\omega} \\
&= \left(\frac{5}{15}\right) e^{(5-15)\frac{1}{6}} \\
&= 0.629
\end{aligned}
$$

(c) Average length of the queue that form from time to time:

$$D = \frac{\mu}{\mu - \lambda} = \frac{15}{15-5}$$
$$= 1.5$$

8) There is congestion on the platform of a railway station. The trains arrive at a rate of 30/days. The service time for any train is ED with an average of 36mins. Calculate:
(a) Mean queue size (2.25)
(b) Probability that there are more than 10 trains in the system. (0.0422)

Solution

Mean arrival rate $= \lambda = \quad 30/days \quad = \frac{30}{24} = 1.25/hr$

Mean service rate $= \mu = \frac{1}{36} \times 60 = 1.66/hr$

(a) Mean queue size:-

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} \quad = \frac{1.25^2}{1.66(1.66 - 1.25)}$$
$$L_q = 2.295 \ per\ hr$$

(b) Probability that than 10 trains in the system n = 11

$$P_{ro} = \left(\frac{\lambda}{\mu}\right)^n = \left(\frac{1.25}{1.66}\right)^{11}$$
$$= 0.0422$$

9) The arrival rate for a waiting line system obeys a P.D with a mean of 0.5 units/hr. it is required that the probability of one or more units in the system does not exceed 0.25. what is the minimum service rate that must be provided if the service duration will be distributed exponentially? (2/hr)

Solution

$$\lambda = 0.5\ units/hr$$

$$P_n\ or\ more = 0.25 \quad = \left(\frac{\lambda}{\mu}\right)^n$$

$$n = 1$$

$$\therefore \frac{0.5}{\mu} = 0.25 \qquad \therefore\ \mu = 2/hr$$

10) In a municipality hospital patients arrival are considered to be Poisson with an arrival interval time of 10mins. The doctors (examination and dispensing) time many be assumed to be ED with an average of 6mins find :

a) What is the chance that a new patient directly sees the doctor? (0.4)

b) For what proportion of the time the doctor is busy? (0.6)

c) What is the average number of patients in the system? (1.5)

d) What is the average waiting time of the system? (1.5)

e) Suppose the municipality wants to recruit another doctor, when an average waiting time of an arrival is 30mins in the queue. Find out hose large should be to justify a 2nd doctor? (λ = 8.33)

Solution

$$\lambda = \frac{1}{10} \times 60 = 6/hr$$

$$\mu = \frac{1}{6} \times 60 = 10/hr$$

(a) Probability that a new patient straight away sees the doctor:-

$$P_0 = 1 - \frac{\lambda}{\mu} \qquad = 1 - \frac{6}{10} \qquad = 0.4$$

(b) Proportion of time the doctor is busy:-

$$\rho = \frac{\lambda}{\mu} \qquad = \frac{6}{10} \qquad = 0.6hr$$

(c) Average number of patients in the system

$$L = L_q + \frac{\lambda}{\mu}$$

$$= \frac{\lambda^2}{\mu(\mu - \lambda)} + \frac{\lambda}{\mu} \qquad = \frac{6^2}{10(10-6)} + \frac{6}{10} \qquad L = 1.5$$

(d) Average waiting in the system:-

$$W = W_q + \frac{1}{\mu} \qquad = \frac{L_q}{\lambda} + \frac{1}{\mu}$$

$$= \frac{\lambda^2}{\mu(\mu - \lambda)\lambda} + \frac{1}{\mu}$$

$$= \frac{6}{10(10-6)} + \frac{1}{10} \qquad W = 0.25$$

$$W_q = \frac{30}{60} \qquad = 05hr$$

(e)

$$W_q = \frac{L_q}{\lambda} \qquad = \frac{\lambda^2}{\mu(\mu-\lambda)\lambda} \qquad = 0.5$$

$$= \frac{\lambda}{10(10-\lambda)} \quad = 0.5 \qquad \therefore \quad \lambda = \frac{50}{6} \quad = 8.33/hr$$

The value of λ has to be increased from 6 to 8.33 justify a second doctor.

11) At a one man barber shop customers arrive according to P.D with a mean arrival rate of 5/hr. The hair cutting time is ED with a hair cut taking 10 min on an average assuming that the customers are always willing to wait find:

a) Average number of customer in the shop [5]

b) Average waiting time of a customer [0.833]

c) The percent of time an arrival Can walkright with out having to wait [16.66%]

d) The probability of a customer waiting more than 5mins [0.766]

Solution

$$\lambda = 5/hr$$

Mean service rate=
$$\mu = \frac{1}{10}\times 60$$
$$= 6/hr$$

(a) Average number of customer's in the shop.

$$L = L_q + \frac{\lambda}{\mu}$$
$$= \frac{\lambda^2}{\mu(\mu-\lambda)} + \frac{\lambda}{\mu}$$
$$= \frac{5^2}{6(6-5)} + \frac{5}{6}$$

L= 5 customers

(b) Average waiting time of a customer.

$$W_q = \frac{L_q}{\lambda}$$
$$= \frac{\lambda^2}{\mu(\mu-\lambda)\lambda}$$
$$= \frac{5}{6(6-5)} \qquad = 0.833hr$$

(c) Percent of time arrival can walk right without having to wait.

$$p_o = \left(1 - \frac{\lambda}{q}\right) \times 100$$
$$= 1 - \frac{5}{6} \times 100$$
$$= 16.66\%$$

d) Probability of a customer waiting more than 5mins.

$$W = \frac{5}{60} \quad = 1/12$$

$$\therefore P_{ro} = \left(\frac{\lambda}{\mu}\right)^{e^{(\lambda-\mu)w}}$$

$$= \left(\frac{5}{6}\right)^{e^{(5-6)1/2}}$$

$$\therefore P_{ro} = 0.766$$

12) At a stamp vender window of a post office 20 customers arrive on an average every 10 min. the vender clerk can serve 5 customers in 2 min. Determine

a)Average number of customer in the System [4]

b) Average waiting time of a customer [0.0266]

c) Probability of a customer waiting more than 3mins before being served[0.1785]

d) Idle time of the vender clerk in a shift of 8hrs [1.6]

$$\lambda = \frac{20}{10}\times 60 \qquad = 120/hr$$

Solution

$$\mu = \frac{5}{2} \qquad = 2.5/\min \quad = 2.5\times 60 \quad = 150/hr$$

(a) Average number of customers in the system:-

$$L = L_q + \frac{\lambda}{\mu} \qquad = \frac{\lambda^2}{\mu(\mu-\lambda)} + \frac{\lambda}{\mu}$$

$$= \frac{120^2}{150(150-120)} + \frac{120}{150} \qquad = 4\ customers$$

(b) Average waiting time of a customer:-.

$$W_q = \frac{L_q}{\lambda} \qquad = \frac{3.2}{120} \qquad = 0.028hr$$

(e) Probability of a customer waiting more than 3mins before being received.

$$W = \frac{3}{60} \qquad = 1/20$$

$$P_{ro} = \left(\frac{\lambda}{\mu}\right)^{e^{(\lambda-\mu)w}} \qquad = \left(\frac{120}{150}\right)^{e^{(120-150)1/20}} \qquad P_{ro} = 0.1785$$

(d) Idle time of the vendor clerk in a shift of 8hours.

$$\begin{aligned} &= P_o \times 8 \\ &= \left(1 - \frac{\lambda}{\mu}\right) \times 8 \\ &= \left(1 - \frac{120}{150}\right) \times 8 \\ &= 1.6hr \end{aligned}$$

13) Arrivals of machinist at a tool crib are considered to be P.D at an average rate of 6/hr. the length of time the machinist must remain at the tool crib is ED with an average time being 0,05 hrs.

a) what is the problem that the machinist arriving at the tool crib will have to wait?
[0.3]
b) What is the average number of machinist at the tool crib? [0.428]
c) The company will install a 2nd tool crib when convinced that a machinist would expect to spent at least 6mins waiting and being served at the tool crib. By how much the flow of machinist to the tool crib increase to justify the 2nd tool crib.
[x = 10]

Solution

$$\lambda = 6/hr$$

$$\mu = \frac{1}{0.05} \qquad = 20/hr$$

(a) Probability that the machinist arriving at the tool crib will have to wait.

$$= 1 - P_o$$

$$= \frac{\lambda}{\mu} \quad = \frac{6}{20} \quad = 0.3$$

(b) Average number of machinist at the tool crib.

$$L = L_q + \frac{\lambda}{\mu}$$

$$= \frac{\lambda^2}{\mu(\mu - \lambda)} + \frac{\lambda}{\mu}$$

$$= \frac{6^2}{20(20-6)} + 0.3$$

$$= 0.42$$

(C)

$$W = \frac{6}{60} \quad = 0.1hr$$

$$0.1 = \quad W = \frac{\lambda}{\mu(\mu - \lambda)} + \frac{1}{\mu}$$

$$0.1 = \frac{\lambda}{20(20-\lambda)} + \frac{1}{20}$$

$$\therefore \lambda = 10/hr$$

14) Jobs arrive at an inspection station according to Poisson process at a mean rate of 2/hr and are inspect one at a time on a FIFO basis. The quality control engineer both inspects and makes minor adjustments. The total service time for the job appears to be ED with a mean of 25mins. Jobs that arrive but cannot be inspected immediately by the engineer must be stored until the engineer is free to take them. Each job requires 1 sq mts space determine

a) The waiting line length [4.16]
b) The waiting time [2.08]
c) % of idle time of the engineer [16.66%]
d) The floor space to be provided in the quality control room. [5]

Solution

$$\lambda = 2 / hr$$

$$\mu = \frac{1}{25} \times 60 = \frac{2.4 / hr}{}$$

(a) $$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} \quad = \frac{2^2}{2.4(2.4 - 2)} \quad = 4.16$$

(b) $$W_q = \frac{L_q}{\lambda} = \quad \frac{4.16}{2} \quad = 2.08$$

c) Idle time of the engineer:-

$$P_o = 1 - \frac{\lambda}{\mu} \times 100 \quad = 16.66\ \%$$

(d) Floor space to be provided in the quality control room

$$L = L_q + \frac{\lambda}{\mu} \qquad = 4.16 + \frac{2}{2.4}$$

$$L = \quad 4.993$$

$$L \cong \quad 5 \times 1 m^2$$

15) The arrival of aircraft at an international tends to follow a Poisson fashion, in spite of schedule flight time, due to high operating variability in the schedule time. It can be assumed that the aircraft arrives at an average rate of 6/hr. The landing service is provided through a single runway by a control tower according to ED with an average service time of 6mins/flight:

(a) Find the prob. that will more than 10mins all together to wait for landing and to land an air-craft. [0.513]

(b) What is prob. that the runway will be free for an incoming flight? [0.4]

Solution

$$\lambda = 6/hr$$

$$\mu = \frac{1}{6} \times 60 \quad = 10/hr$$

$$V = \frac{10}{60} \quad = \frac{1}{6}$$

$$P_{ro} = e^{(\lambda-\mu)v}$$

(a) $$= e^{(6-10)1/6}$$

$$= 0.513$$

b) Probability that the runway will be free for an incoming flight.

$$P_o = 1 - \frac{\lambda}{\mu}$$

$$= 1 - \frac{6}{10}$$

$$P_o \quad = 0.4$$

16) At what rate must the clerk of a super market work in order to ensure a prob. Of 0.9 that the Customer will not have to wait longer than 12mins in the system. It is assumed that the arrivals follows a Poisson fashion at the rate of 15/hr. The length of service by the clerk has an ED.

(a) Also find the average number of customers queuing for service. [0.738]

(b) The Prob. of having more than 10 customers in the system [1.9×10^{-3}]

Solution:

$$v = \frac{12}{60} = \frac{1}{5} = 0.2$$

$$\lambda = 15 / hr$$

$$P_{ro} = e^{(\lambda - \mu)}$$

$$1 - 0.9 = e^{(15-\mu)0.2}$$

$$0.1 = e^{(15-\mu)}$$

$$\therefore \quad \mu 0.1 = (15 - \mu)0.2$$

$$\therefore \quad \mu = 26.5$$

(a) Average number of customer's queuing for service

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

$$= \frac{15^2}{26.5(26.5 - 15)}$$

$$L_q = 0.738$$

(b) Probability of having more than 10 customers in the system:

$$n = 11$$

$$P_{ro} = \left(\frac{\lambda}{\mu}\right)^n$$

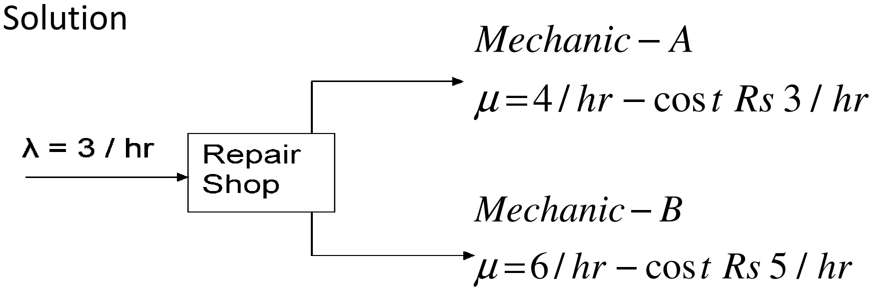
$$P_{ro} = \left(\frac{15}{26.5}\right)^{11}$$

$$P_{ro} = 0.00191$$

17) A mechanic is to hired to repair a machine which breaks down at an average rate of 3/hr. breakdowns are distributed in time in a manner that may be regarded as Poisson. The non-productive time on any machine is considered to cost the Company. Rs. 5/hr. The Company has the choice to 2 mechanics A & B.

The mechanic A repairs the machines at an average rate of 4/hr and he will demand Rs. 3/hr. The mechanic B costs Rs. 5/hr and can repair the machines exponentially at an average rate of 6/hr. Decide which mechanic should be hired.

Solution



Consider the mechanic A

$\lambda = 3 / hr, \quad \mu = 4 / hr$

The number of break down machine in the system

$$L = L_q + \frac{\lambda}{\mu}$$

$$= \frac{\lambda^2}{\mu(\mu-\lambda)} + \frac{\lambda}{\mu}$$

$$= \frac{3^2}{4(4-3)} + \frac{3}{4}$$

$$L = 3 \ m/c's$$

The non-productive time the company / hr = 3 x 5 =15 Rs

The amount paid to the mechanic A per hour = 3 Rs

The total expected cost per hour = 15+3 = 18 Rs

Consider the mechanic B

λ = 3 / hr, μ = 6 / hr

The number of break down machine in the system

$$L = L_q + \frac{\lambda}{\mu}$$

$$= \frac{\lambda^2}{\mu(\mu-\lambda)} + \frac{\lambda}{\mu}$$

$$= \frac{3^2}{6(6-3)} + \frac{3}{6}$$

$$L = 1 \ m/c's$$

The non-productive time cost of the company / hr = 1 x 5 = 5 Rs

Amount paid to the mechanic / hr = 5

Total expected cost per hour = 5+5 = 10 Rs

Selected mechanic B as the total expected cost per hour of mechanic B is less than the total expected cost per hour of mechanic A.

QUEUEING THEORY PROBLEMS : MODEL – 2

Multiple server Model

M / M / K : ∞ / FCFS

Formulae

Multiple server Model

(1)

$$P_o = \frac{1}{\left[\sum_{n=0}^{K-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{K!}\left(\frac{\lambda}{\mu}\right)^K \frac{k\mu}{(k\mu - \lambda)}}$$

(2)

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n . P_o$$

(3)

$$L_q = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^K}{(k-1)!(k\mu - \lambda)^2} \times P_o$$

(4)

$$L = L_q + \frac{\lambda}{\mu}$$

(5)

$$W_q = \frac{L_q}{\lambda}$$

(6)

$$W = Wq + \frac{1}{\mu}$$

(7)

$$\rho = \left(\frac{\lambda}{k\,\mu}\right)$$

1) A Commercial bank has 3 cash paying assistants customers are found to arrive in a Poisson fashion at an average rate of 6/hr for business transaction. The service time is found to have an E.D with a mean of 18 mins. The customers are processed on FCFS basis. Calculate

a) Average number of customers in the system

b) Average time a customer spends in the system

c) Average queue length

d) How many hours a week can a cash paying assistant spend with the customers.

Solution:

$$K = 3$$

$$\lambda = 6/hr$$

$$\mu = \frac{1}{18} \times 60 = 3.33/hr$$

$$P_o = \frac{1}{\left[\sum_{n=0}^{K-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{K!}\left(\frac{\lambda}{\mu}\right)^K \frac{k\mu}{k\mu - \lambda}}$$

$$= \frac{1}{\frac{1}{0!}\left(\frac{6}{3.33}\right)^0 + \frac{1}{1!}\left(\frac{6}{3.33}\right)^1 + \frac{1}{2!}\left(\frac{6}{3.33}\right)^2 + \frac{1}{3!}\left(\frac{6}{3.33}\right)^3 \frac{3\times3.33}{3\times3.33 - 6}}$$

$$P_o = 0.145$$

(a) Average number customers in the system:-

$$L_q = \frac{\lambda \mu \left(\frac{\lambda}{\mu} \right)^K}{(k-1)!(k\mu - \lambda)^2} \times P_o$$

$$= \frac{6 \times 3.33 \left(\frac{6}{3.33} \right)^3}{2!(3 \times 3.33 - 6)^2} \times 0.145$$

$$L_q = 0.532$$

$$\therefore \; L = L_q + \frac{\lambda}{\mu}$$

$$= 0.532 + \frac{6}{3.33}$$

$$L = 2.334$$

(b) *Average time a customer spends in the system*

$$W = Wq + \frac{1}{\mu} \quad = \frac{Lq}{\lambda} + \frac{1}{\mu}$$

$$= \frac{0.532}{6} + \frac{1}{3.33} \quad = \; 0.388$$

(c) *Average queue length*

$$Lq = 0.532$$

(d) Assuming 5 days a week and 8 hrs a day the number of hrs in a week the cash paying assistant spends with the customers

$$= \rho \times 5 \times 8$$

$$= \left(\frac{\lambda}{k\,\mu}\right) \times 5 \times 8$$

$$= \frac{6}{3 \times 3.33} \times 5 \times 8$$

$$= 24.02\, hrs$$

(2) A telephone exchange has two long distant operators. The telephone company finds that during the fashion at an average rate of 15/hr. The length mean of 5 mins.

(a) what is the probability that a subscriber will have to wait for his long distant dials on the peak hour of the day.

(b) what is the average waiting time for the customers.

Solution:

$$K = 2$$

$$\lambda = 15 / hr$$

$$\mu = \frac{1}{5} \times 60 = 12 / hr$$

$$P_o = \frac{1}{\frac{1}{0!}\left(\frac{15}{12}\right)^0 + \frac{1}{1!}\left(\frac{15}{12}\right)^1 + \frac{1}{2!}\left(\frac{15}{12}\right)^2 \frac{2 \times 12}{2 \times 12 - 15}}$$

$$P_o = 0.230$$

(a) Probability that a subscriber will have to wait for his long distant call is

$$P_{ro} = 1 - P_o - P_1$$

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n .P_o$$

$$\therefore \quad P_1 = \frac{1}{1!}\left(\frac{15}{12}\right)^1 \times 0.230$$

$$P_1 = 0.2875$$

$$\therefore \qquad = 1 - 0.230 - 0.2875$$

$$P_{ro} = 0.48$$

(b) average waiting time for the customer

$$W_q = \frac{L_q}{\lambda}$$

$$= \frac{\lambda\,\mu\left(\frac{\lambda}{\mu}\right)^k}{\lambda\,(k-1)!(k\,\mu - \lambda)^2} \times P_o$$

$$= \frac{12\left(\frac{15}{12}\right)^2 \times 0.23}{1!(24-15)^2} = 0.053\,hr$$

3) An insurance company has 3 clerks in its branch office. People arrive with claims against the company are found to arrive in a Poisson fashion at an average of 20 per 8 hours a day. The amount of time that a clerk spends with the client is found to have ED with a mean time of 40 mins. The clients are processed in the order of their appearance.

(a) How many hours a week can a clerk expect to spend with the clients.

(b) How much time an average does a client spend in the branch office.

Solution:

$$K = 3$$

$$\lambda = \frac{20}{8} = 2.5 / hr$$

$$\mu = \frac{1}{40} \times 60 = 1.5 / hr$$

$$P_o = \frac{1}{\left[\sum_{n=0}^{K-1} \frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n \right] + \frac{1}{K!} \left(\frac{\lambda}{\mu} \right)^K \frac{k\mu}{k\mu - \lambda}}$$

$$= \frac{1}{\frac{1}{0!} \left(\frac{2.5}{1.5} \right)^0 + \frac{1}{1!} \left(\frac{2.5}{1.5} \right)^1 + \frac{1}{2!} \left(\frac{2.5}{1.5} \right)^2 + \frac{1}{3!} \left(\frac{2.5}{1.5} \right)^3 \frac{3 \times 1.5}{3 \times 1.5 - 2.5}}$$

$$P_o = 0.173$$

(a) The number of hours per week a clerk expects to spend with the client

= ρ x 5 x 8 assuming 5 days a week and 8 hrs / day

$$= \frac{\lambda}{\mu k} \times 5 \times 8$$

$$= \frac{2.5}{1.5 \times 3} \times 5 \times 8 \quad = 22.22\ hrs$$

(b) *Average time a clerk spends in the branch office*

$$W = Wq + \frac{1}{\mu}$$

$$= \frac{Lq}{\lambda} + \frac{1}{\mu}$$

$$= \frac{\dfrac{\lambda\,\mu\left(\dfrac{\lambda}{\mu}\right)^{k} \times P_o}{(k-1)!\,(k\,\mu-\lambda)^2}}{\lambda} + \frac{1}{\mu}$$

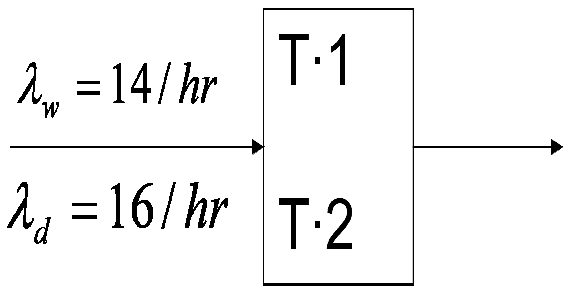
$$= \frac{1.5\left(\dfrac{2.5}{1.5}\right)^{3} \times 0.173}{2\,!(3\times 1.5-2.5)^2} + \frac{1}{1.5}$$

$$= 0.816\ hrs.$$

4) A bank has 2 tellers working on saving accounts. The 1st tellers handles withdrawal's only and the 2nd teller handles deposits only, it has been found that service time distribution for depositors and withdrawal's. Both are E.D with a mean service time of 3min /customer. Depositor are found to arrive in a Poisson fashion with a arrival rate of 16/hr and withdrawal's also drive in a Poisson with a mean rate of 14/hr.

What would be the effect on the average waiting time for the depositors and withdrawal's if each teller would handle both withdrawal's and deposits. What would be the effect if the time would only be accomplished by increasing the service time to 3.5mins.

Solution:



Waiting time in the queue for the depositors

$$Wq_D = \frac{L_q}{\lambda}$$

$$= \frac{\lambda^2}{\lambda\mu\,(\mu - \lambda)} \quad = \frac{16}{20(20-16)} \quad = 0.2hrs$$

waiting time in the queue for the withdrawal's

$$Wq_w = \frac{L_q}{\lambda} \quad = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$= \frac{14}{20(20-14)} \quad = 0.1160hrs$$

Consider it as a multi server model.

K = 2

$$\lambda = \lambda_W + \lambda_D = 30 / hr$$

$$\mu = 20 / hr$$

$$P_o = \frac{1}{\frac{1}{o!}\left(\frac{30}{20}\right)^0 + \frac{1}{1!}\left(\frac{30}{20}\right)^1 + \frac{1}{2!}\left(\frac{30}{20}\right)^2 \frac{2\times 20}{2\times 20-30}}$$

$$P_o = 0.143$$

Waiting time in the queue

$$w_q = \frac{L_q}{\lambda}$$

$$= \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^k \times P_o}{\lambda\,(K-1)!(K\mu-\lambda)^2} \qquad = \frac{20\left(\frac{30}{20}\right)^2 \times 0.143}{1!\,(2\times 20-30)^2}$$

= 0.064hrs is the waiting time in the queue

If the service time is increased from 3 to 3.5mins

$$P_o = \frac{1}{\frac{1}{0!}\left(\frac{30}{17.14}\right)^0 + \frac{1}{1!}\left(\frac{30}{17.14}\right)^1 + \frac{1}{2!}\left(\frac{30}{17.14}\right)^2 \frac{2\times 17.14}{2\times 17.14\_30}}$$

$$P_0 = 0.065$$

$$W_q = \frac{17.14\left(\frac{30}{17.14}\right)^2 \times 0.065}{1!\,(2\times 17.14-30)^2}$$

$$W_q = 0.192hrs$$

By increasing the service time from 3 to 3.5 min the waiting time in the queue for the depositors is decreased from 0.2 to 0.19hrs.

But in the case of withdrawal's the waiting time in the queue increased from 0.116hrs to 0.19hrs

5) A tax consulting firm has 3 counters in its offices to receive the people who have problems concerning their income and the sales tax. On an average 48 persons arrive in 8hrs a day. Each tax advisor spends 15 min an on average for a arrival of the arrival time follows a Poisson distribution and the service time follows a E.D.

(a) Find the average number of customer in the system.

(b) Average waiting time of the customer in the system.

(c) Average number of customers waiting the queue for service.

(d) Average waiting time of the customers in the queue.

(e) How many hours each week a tax advisor spends performing his job.

(f) Probability that a customer has to wait before he gets service.

(g) Expected number of idle tax advisors at any specified time

Solution:

$$K = 3$$

$$\lambda = \frac{48}{8} \quad = 6hrs$$

$$\mu = \frac{1}{15} \times 60 \quad 4/hrs$$

$$P_0 = \frac{1}{\frac{1}{0!}\left(\frac{6}{4}\right)^0 + \frac{1}{1!}\left(\frac{6}{4}\right)^1 + \frac{1}{2!}\left(\frac{6}{4}\right)^2 + \frac{1}{3!}\left(\frac{6}{4}\right)^3 \frac{3\times4}{3\times4-6}}$$

$$\therefore P_0 = \quad 0.210$$

a) Average number of customers in the system:

$$L = L_q + \frac{\lambda}{\mu}$$

$$= \frac{\lambda\mu(\lambda/\mu)^K}{(K-1)!(K\mu-\lambda)^2} \times P_0 + \frac{\lambda}{\mu}$$

$$= \frac{6\times4\left(\frac{6}{4}\right)^3 \times 0.21}{2!(3\times4-6)^2} + \frac{6}{4} \qquad L = 1.73\, customers$$

b) Average waiting time of the customer in the system.

$$W = W_q + \frac{1}{\mu}$$

$$= \frac{L_q}{\mu} + \frac{1}{\mu}$$

$$= \frac{\lambda\mu(\lambda/\mu)^K \times P_0}{\lambda(K-1)!(K\mu-\lambda)^2} + \frac{1}{\mu}$$

$$= \frac{4\left(\frac{6}{4}\right)^3 \times 0.21}{2!(3\times4-6)^2} + \frac{1}{4}$$

$$W = 0.289\, hour$$

(c) Average number of customer in the queue

$$L_q = \frac{\lambda\mu(\lambda/\mu)^K \times P_0}{(K-1)!(3\times4-6)}$$

$$L_q = 0.23$$

(d) Average time of the customers in the queue.

$$W_q = \frac{L_q}{\lambda} \quad = \frac{0.23}{6} \quad = 0.038$$

(e) Assuming 5 days a week and 8 hours per day the number of hours the tax advisors spents with customers during the week.

$$= \rho \times 5 \times 8$$

$$= \frac{\lambda}{K\mu} \times 5 \times 8 \qquad = \frac{6}{3 \times 4} \times 5 \times 2$$

$$= 5 hours$$

(f) Probability that a customer has to wait before he gets service

$$= 1 - P_0 - P_1 - P_2$$

$$P_1 = \frac{1}{1!}\left(\frac{\lambda}{\mu K}\right)^1 \times P_0 = 0.105$$

$$= \frac{1}{11}\left(\frac{6}{3 \times 4}\right)^1 \times 0.21 \qquad = 0.02625$$

$$P_2 = \frac{1}{2\,!}\left(\frac{6}{3 \times 9}\right)^2 \times 0.21$$

$$= 0.02625$$

$$\therefore 1 - 0.21 - 0.105 - 0.02625$$

$$= 0.239$$

g) Expected number of idle tax advisors at any specified time

$$= 3P_0 + 2P_1 + 1P_2 + 0P_3$$
$$= 3\times 0.21 + 2\times 0.105 + 1\times 0.02625$$
$$= \quad 1.5\ \%$$

Reference Books:

1. **Taha H A**, Operation Research - An Introduction, Prentice Hall of India, 7th edition, 2003
2. **Ravindran, Phillips and Solberg**, Operations Research : Principles and Practice, John Wiely & Sons, 2nd Edition
3. **D.S.Hira**, Operation Research, S.Chand & Company Ltd., New Delhi, 2004