ANALYSIS OF TRUSSES

A Truss is a two force members made up of bars that are connected at the ends by joints. Every stress element is in either tension or compression. Trusses can be classified as plane truss and space truss.

Plane truss is one where the plane of the structure remain in plane even after the application of loads

While space truss plane will not be in a same plane

Fig shows 2d truss structure and each node has two degrees of freedom. The only difference between bar element and truss element is that in bars both local and global coordinate systems are same where in truss these are different.

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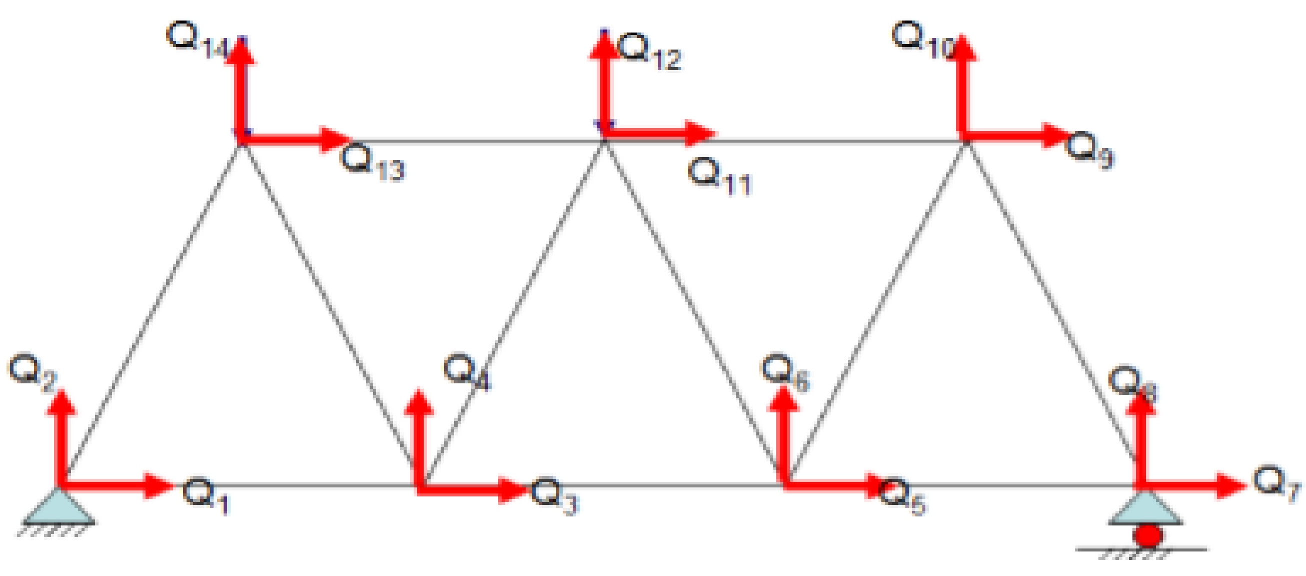
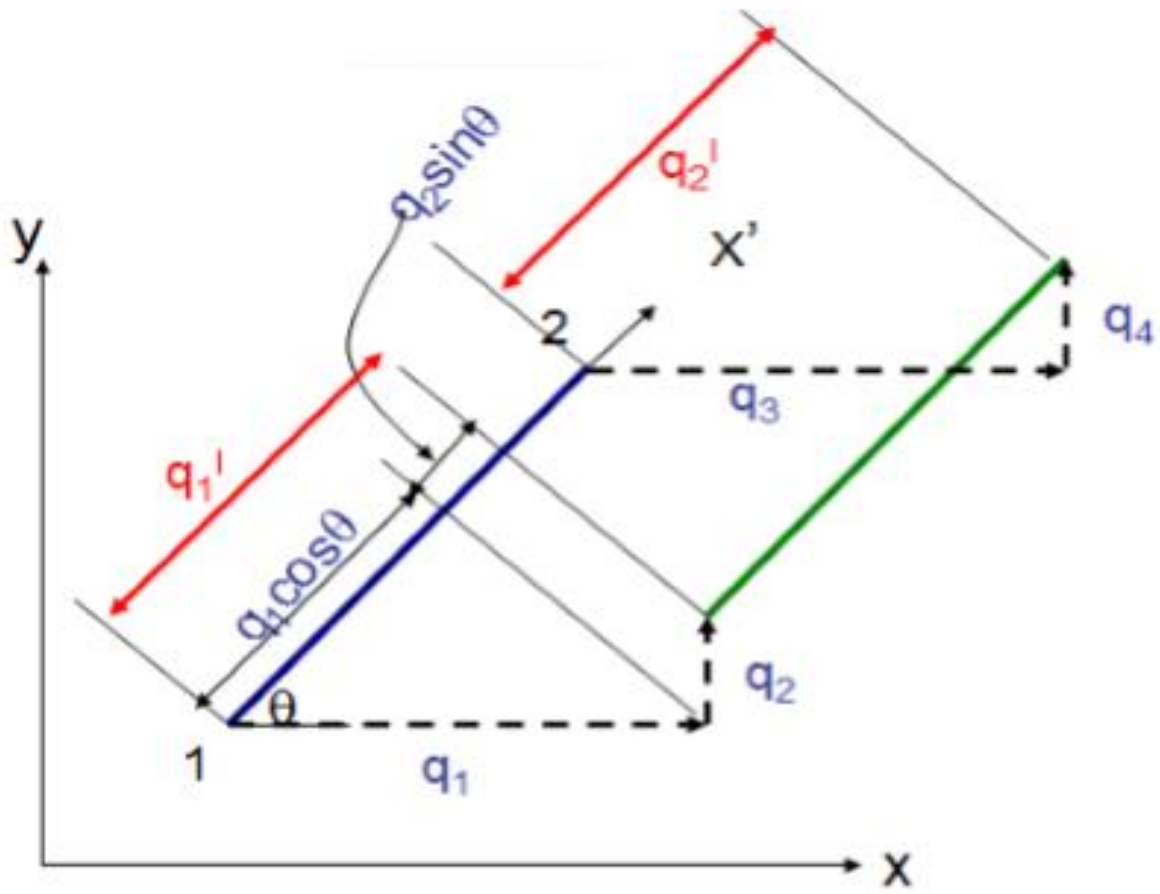


There are always assumptions associated with every finite element analysis. If all the assumptions below are all valid for a given situation, then truss element will yield an exact solution. Some of the assumptions are:

- Truss element is only a prismatic member ie cross sectional area is uniform along its length
- It should be a isotropic material
- Constant load i.e load is independent of time
- Homogenous material
- A load on a truss can only be applied at the joints (nodes)
- Due to the load applied each bar of a truss is either induced with tensile/compressive forces
- The joints in a truss are assumed to be frictionless pin joints
- Self weight of the bars are neglected

Derive the element Stiffness matrix for Truss

Consider one truss element as shown that has nodes 1 & 2



The coordinate system that passes along the element (X' axis) is called local coordinate and X-Y system is called as global coordinate system.

After the loads applied let the element takes new position say locally node 1 has displaced by an amount q1' and node2 has moved by an amount equal to q2'.

As each node has 2 dof in global coordinate system .let node 1 has displacements q1 and q2 along x and y axis respectively similarly q3 and q4 at node 2.

Resolving the components q1, q2, q3 and q4 along the bar we get two equations as

$$q_1^{l} = q_1 \cos\theta + q_2 \sin\theta$$

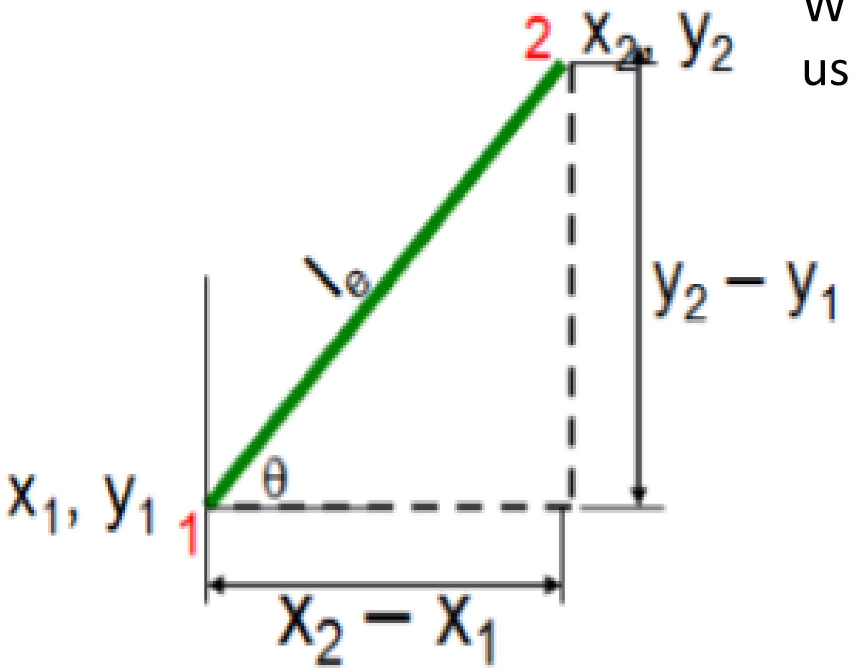
$$q_2^{l} = q_3 \cos\theta + q_4 \sin\theta$$

Also written using Direction Cosines

$$q_1' = q_1 \ell + q_2 m$$
$$q_2' = q_3 \ell + q_4 m$$

How to calculate direction cosines

Consider a element that has node 1 and node 2 inclined by an angle as shown . let (x1, y1) be the coordinate of node 1 and (x2,y2) be the coordinates at node 2.



When orientation of an element is know we use this angle to calculate l and m as:

$$\ell = \cos\theta$$
$$m = \cos(90 - \theta) = \sin\theta$$

and by using nodal coordinates we can calculate using the relation

$$\ell = \frac{x_2 - x_1}{l_e} \qquad m = \frac{y_2 - y_1}{l_e}$$

We can calculate length of the element as

$$l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$q_1' = q_1 \ell + q_2 m$$

$$q_2' = q_3 \ell + q_4 m$$

Writing the same equation into the matrix form

$$\begin{pmatrix} q_1' \\ q_2' \end{pmatrix} = \begin{pmatrix} \ell & m & 0 & 0 \\ 0 & 0 & \ell & m \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

$$q' = L\,q$$

Where L is called transformation matrix that is used for local – global correspondence.

Strain energy for a bar element we have

$$U = \frac{1}{2}\, q^T K q$$

Strain energy for a truss element we can write

$$U = \frac{1}{2}\, q^{1T} K\, q^1$$

Where $q^1 = L\, q$ and $q^{1T} = L^T q^T$

Therefore

$$U = \frac{1}{2}\, q^{1T} K\, q^1$$

$$= \frac{1}{2}\, L^T q^T K\, L q$$

$$= \frac{1}{2}\, q^T (L^T K\, L) q$$

$$= \frac{1}{2}\, q^T K_T q$$

Where KT or K is the stiffness matrix of truss element

$$K_T = L^T K\, L$$

$$L = \begin{pmatrix} \ell & m & 0 & 0 \\ 0 & 0 & \ell & m \end{pmatrix}$$

$$L^T = \begin{pmatrix} \ell & 0 \\ m & 0 \\ 0 & \ell \\ 0 & m \end{pmatrix}$$

$$K = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K = \begin{pmatrix} \ell & 0 \\ m & 0 \\ 0 & \ell \\ 0 & m \end{pmatrix} \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} \ell & m & 0 & 0 \\ 0 & 0 & \ell & m \end{pmatrix}$$

Taking the product of all these matrix we have stiffness matrix for truss element which is given as

$$K_T = \frac{AE}{L}\begin{pmatrix} \ell^2 & \ell m & -\ell^2 & -\ell m \\ \ell m & m^2 & -\ell m & -m^2 \\ -\ell^2 & -\ell m & \ell^2 & \ell m \\ -\ell m & -m^2 & \ell m & m^2 \end{pmatrix}$$

The stress σ in a truss element is given by

$$\sigma = \varepsilon E$$

But strain $\varepsilon = B\, q^1$ and $q^1 = T\, q$

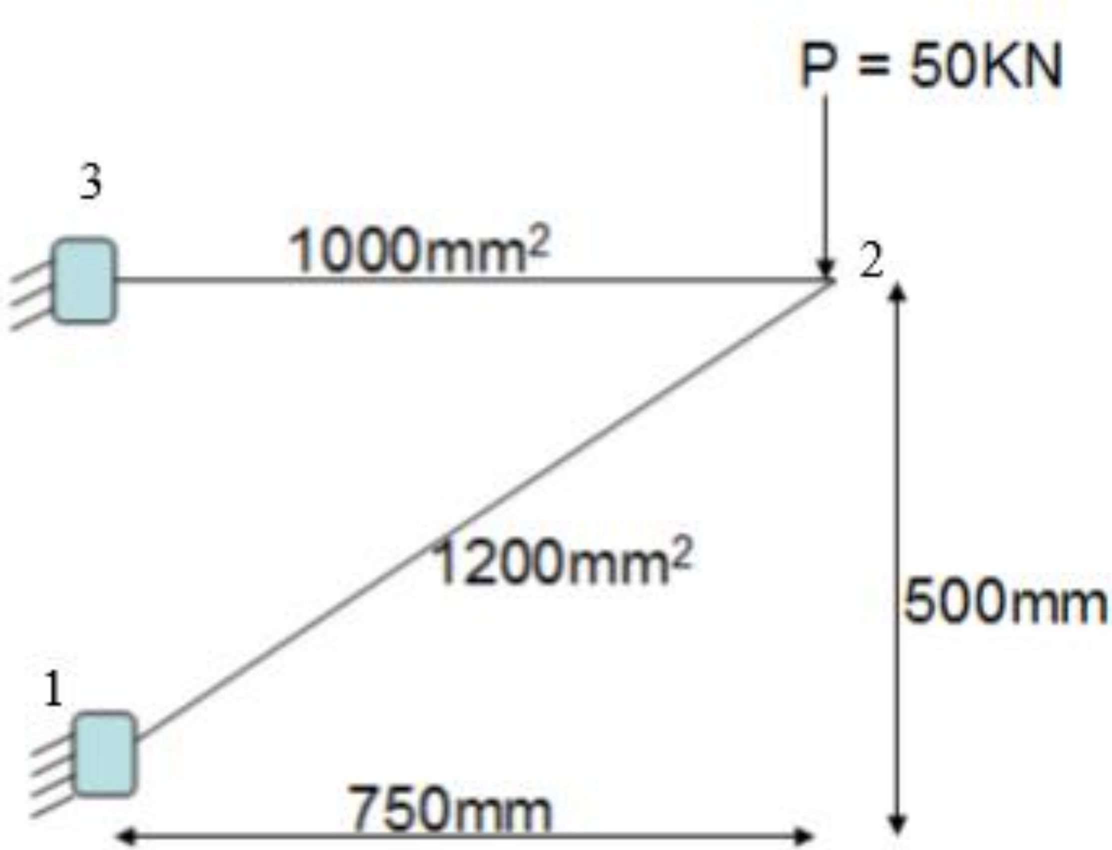
where $B = \frac{1}{L}[-1 \quad 1]$

Therefore

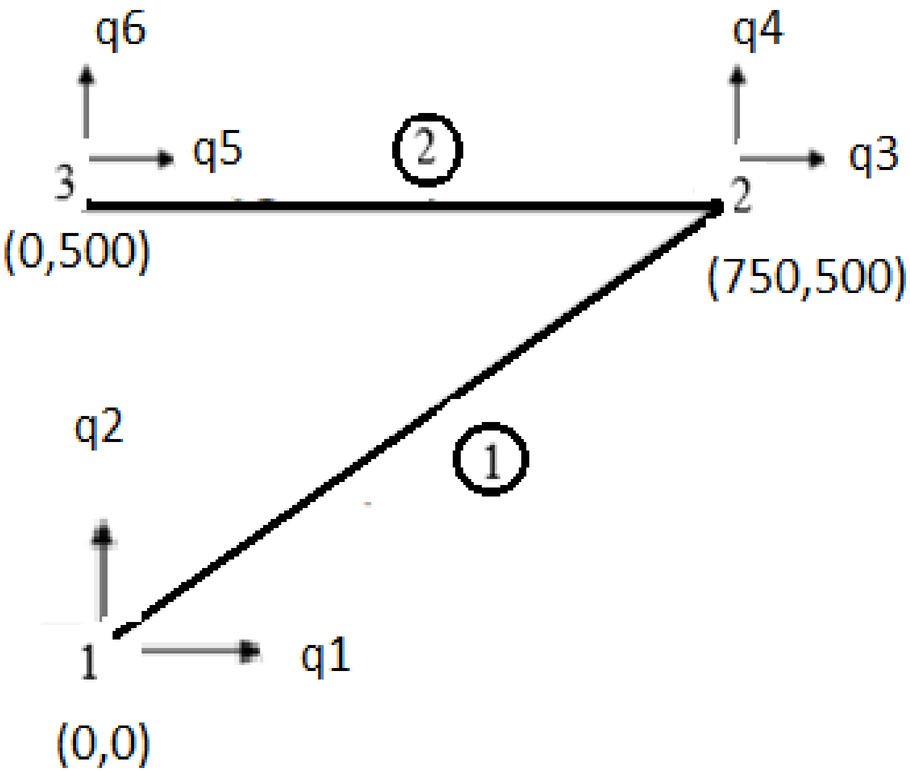
$$\sigma = \frac{E}{L_e}\begin{pmatrix} -\ell & -m & \ell & m \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

Problems on Trusses

For the two bar truss shown in Figure, Determine the nodal displacements and the stress in each element also find support reaction. Take E= 200GPA



Solution: For given structure if node numbering is not given we have to number them which depend on user. Each node has 2 dof say q1 q2 be the displacement at node 1, q3 & q4 be displacement at node 2, q5 & q6 at node 3.



A1 = 1200mm2

A2 = 1000mm2

E = 200GPa = E1 = E2

Nodal coordinate data Table

Node No	x	y
1	0	0
2	750	500
3	0	500

Element Connectivity table

Element	Node NO	Length of the Element (le)	Direction cosines	
			l	m
1	1-2	901.387	0.832	0.554
2	2-3	750	-1	0

Node No	x	y
1	**0**	**0**
2	**750**	**500**
3	**0**	**500**

$$le1 = \sqrt{(x1-x2)^2 + (y1-y2)^2}$$

$$le1 = \sqrt{(0-750)^2 + (0-500)^2} = \mathbf{901.387}$$

$$le2 = \sqrt{(x3-x2)^2 + (y3-y2)^2}$$

$$le2 = \sqrt{(0-750)^2 + (500-500)^2} = \mathbf{750}$$

l1 = (x2-x1)/le1

l1 = (750-0)/901.387 = 0.832

m1 = (y2-y1)/le1

m1 = (500-0)/901.387 = 0.554

l2 = (x3-x2)/le2

l2 = (0 - 750)/750 = -1

m2 = (y3-y1)/le2

m2 = (500 - 500)/750 = 0

Calculate stiffness matrix for both the elements

$$K_T = \frac{AE}{L}\begin{pmatrix} \ell^2 & \ell m & -\ell^2 & -\ell m \\ \ell m & m^2 & -\ell m & -m^2 \\ -\ell^2 & -\ell m & \ell^2 & \ell m \\ -\ell m & -m^2 & \ell m & m^2 \end{pmatrix}$$

$$K_1 = 10^5 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1.84 & 1.22 & -1.84 & -1.22 \\ 1.22 & 0.816 & -1.22 & -0.816 \\ -1.84 & -1.22 & 1.84 & 1.22 \\ -1.22 & -0.816 & 1.22 & 0.816 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$K_2 = 10^5 \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 2.66 & 0 & -2.66 & 0 \\ 0 & 0 & 0 & 0 \\ -2.66 & 0 & 2.66 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Element 1 has displacements q1, q2, q3, q4. Hence numbering scheme for the first stiffness matrix (K1) as 1 2 3 4 similarly for K2 3 4 5 & 6 as shown above.

Global stiffness matrix: the structure has 3 nodes at each node 3 dof hence size of global stiffness matrix will be 3 X 2 = 6

ie 6 X 6

$$K=10^5\;\begin{array}{c}\begin{array}{cccccc}1 & 2 & 3 & 4 & 5 & 6\end{array}\\ \left(\begin{array}{cccccc}1.84 & 1.22 & -1.84 & -1.22 & 0 & 0\\ 1.22 & 0.816 & -1.22 & -0.816 & 0 & 0\\ -1.84 & -1.22 & 4.5 & 1.22 & -2.66 & 0\\ -1.22 & -0.816 & 1.22 & 0.816 & 0 & 0\\ 0 & 0 & -2.66 & 0 & 2.66 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{array}\right)\end{array}\begin{array}{c}1\\2\\3\\4\\5\\6\end{array}$$

$$K_1 = 10^5 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1.84 & 1.22 & -1.84 & -1.22 & 0 & 0 \\ 1.22 & 0.816 & -1.22 & -0.816 & 0 & 0 \\ -1.84 & -1.22 & 1.84 & 1.22 & 0 & 0 \\ -1.22 & -0.816 & 1.22 & 0.816 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$+$$

$$K_2 = 10^5 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2.66 & 0 & -2.66 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2.66 & 0 & 2.66 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

ie 6 X 6

$$K = 10^5 \begin{array}{cc} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} & \\ \begin{bmatrix} 1.84 & 1.22 & -1.84 & -1.22 & 0 & 0 \\ 1.22 & 0.816 & -1.22 & -0.816 & 0 & 0 \\ -1.84 & -1.22 & 4.5 & 1.22 & -2.66 & 0 \\ -1.22 & -0.816 & 1.22 & 0.816 & 0 & 0 \\ 0 & 0 & -2.66 & 0 & 2.66 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \end{array}$$

From the equation KQ = F we have the following matrix. Since node 1 is fixed q1=q2=0 and also at node 3 q5 = q6 = 0 .At node 2 q3 & q4 are free hence has displacements. In the load vector applied force is at node 2 ie F4 = 50KN rest other forces zero.

$$10^5 \begin{pmatrix} 1.84 & 1.22 & -1.84 & -1.22 & 0 & 0 \\ 1.22 & 0.816 & -1.22 & -0.816 & 0 & 0 \\ -1.84 & -1.22 & 4.5 & 1.22 & -2.66 & 0 \\ -1.22 & -0.816 & 1.22 & 0.816 & 0 & 0 \\ 0 & 0 & -2.66 & 0 & 2.66 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Q1 \\ Q2 \\ Q3 \\ Q4 \\ Q5 \\ Q6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -50 \times 10^3 \\ 0 \\ 0 \end{pmatrix}$$

By elimination method the matrix reduces to 2 X 2 and solving we get Q3= 0.28mm and Q4 = -1.03mm. With these displacements we calculate stresses in each element.

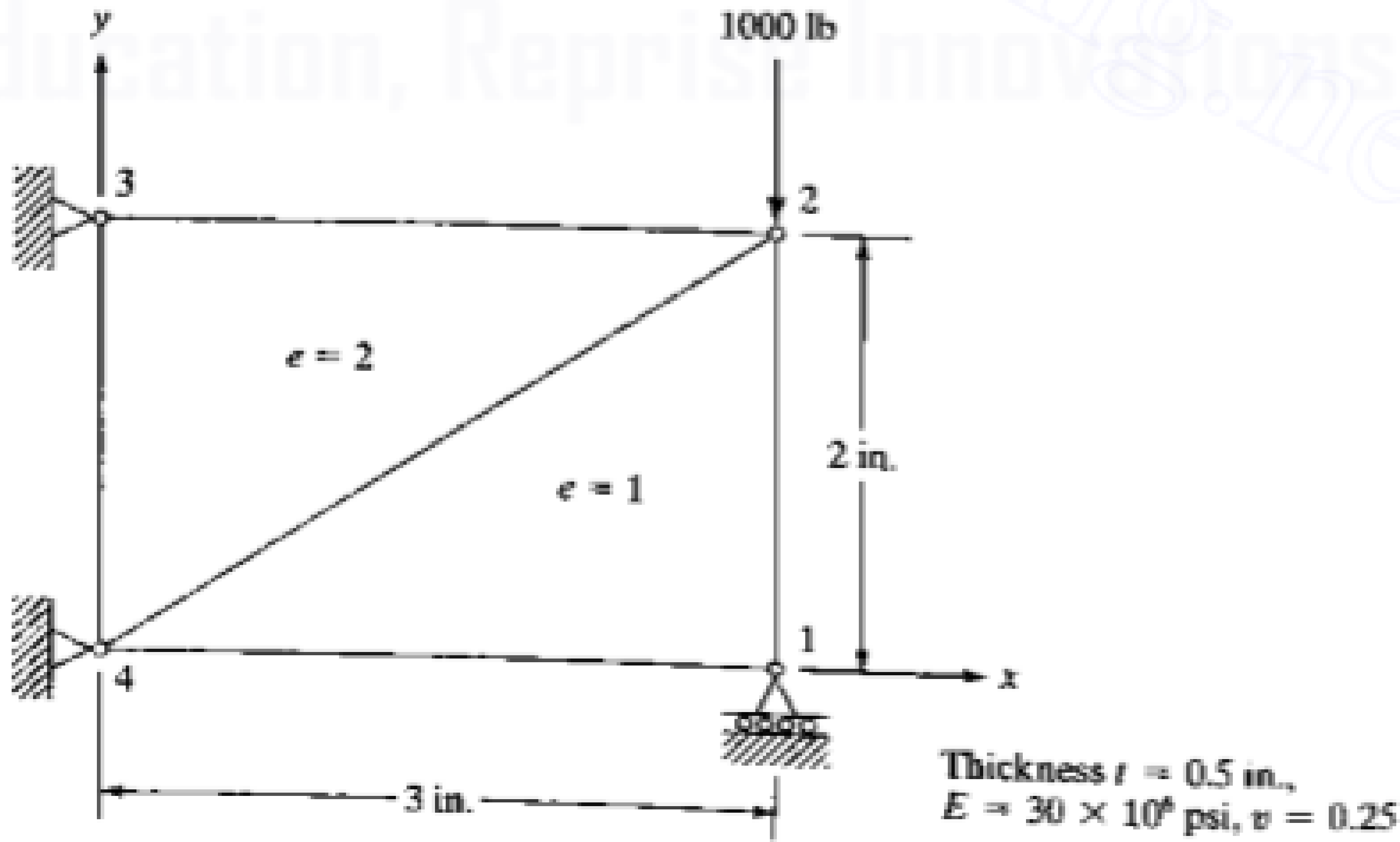
Stress in each Element

$$\sigma = \frac{E}{L_e}\begin{bmatrix} -\ell & -m & \ell & m \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

$$\sigma_1 = \frac{E}{L_{e1}}\begin{bmatrix} -\ell_1 & -m_1 & \ell_1 & m_1 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = -75.32 \text{ N/mm2}$$

$$\sigma_2 = \frac{E}{L_{e2}}\begin{bmatrix} -\ell_2 & -m_2 & \ell_2 & m_2 \end{bmatrix}\begin{Bmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} = 74.64 \text{ N/mm2}$$

For the two-dimensional loaded plate shown in Fig. , determine the displacements of nodes 1 and 2 and the element stresses using plane stress conditions. Body force may be neglected in comparison with the external forces.



FIGURE

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MODULE-2

2. ONE DIMENSIONAL ANALYSIS OF BAR

2.1 * Derivation of Stiffness Matrix For 1-D Bar Element

2.2 * Global Stiffness Matrix.

2.3 * Properties of Global Stiffness Matrix.

2.4 * Problems on 1-D Bar element

2.41 * Problems on Stepped bar Element using Elimination Method.

2.42 * Taper bar problem. For 1-D Element.

2.43 * Bar Problem using penalty approach Method.

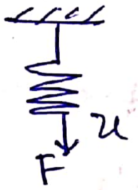
BAR - ELEMENT : (1-D Element)

Consider the finite element Model of bar elements in one-Dimensional problem. Force acting on bar element along the axis. There fore problem is called 1-D problem.

Derive the Stiffness Matrix for 1-D bar element.

Stiffness Matrix for spring element can be defined as load per unit deflection and stiffness are denoted by 'K'

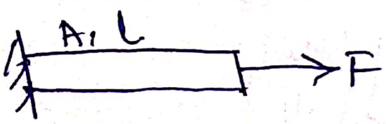
$$K = \frac{load}{deflection}$$



$$K = \frac{F}{u} \Rightarrow K \cdot u = F$$

$$\therefore \boxed{F = K \cdot u} \quad \text{—} (1)$$

For Bar Element we can now define stiffness which can be suitable for bar Element



where $A \rightarrow$ Cross sectional Area, l = length of the bar
$F \rightarrow$ Force applied and $E \rightarrow$ young's modulus.

We Know from Hook's Law

$$\sigma \propto \epsilon \quad \therefore \sigma = E\epsilon$$

Also $\sigma = \frac{F}{A}$ $\quad \epsilon = \frac{\text{change in length}}{\text{original length}} = \frac{u}{L}$

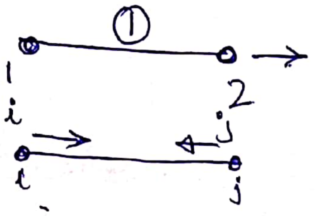
$$\therefore \frac{F}{A} = E \cdot \frac{u}{L}$$ where $u \rightarrow$ displacement.

$$\therefore F = A \cdot E \cdot \frac{u}{L} \qquad F = \frac{AE}{L} \cdot u \quad \text{— (2)}$$

Compare equation (1) & (2)

$$\therefore K = \frac{AE}{L}$$. Mathematically

Let us one 1-D element have 2-Nodes.



Step 1: Consider node 1 at i, some forces F_i is acting and at node 2 at j reaction forces will exist

$$\therefore F_i = \frac{AE}{L} \cdot u_i \quad \text{— (1)} \qquad F_j = -\frac{AE}{L} \cdot u_i \quad \text{— (2)}$$

$$F_i = -F_j$$

Step 2: Consider at node 2 at j some is acting and node 1 is fixed.

$$\therefore F_j = \frac{AE}{L} \cdot u_j \quad \text{— (3)} \qquad F_i = -F_j$$

$$F_i = -\frac{AE}{L} u_j \quad \text{— (4)}$$

writing All 4 equations together.

$$F_i = \frac{AE}{L} u_i \qquad F_j = -\frac{AE}{L} u_i$$

$$F_i = -\frac{AE}{L} u_j \qquad F_j = \frac{AE}{L} \cdot u_j$$

$$F_i = \left[\frac{AE}{L} u_i \quad -\frac{AE}{L} u_j\right] ; \quad F_j = \left[-\frac{AE}{L} u_i \quad \frac{AE}{L} u_j\right]$$

$$\begin{Bmatrix} F_i \\ F_j \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} u_i & -u_j \\ -u_i & u_j \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

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Properties of The Stiffness Matrix.

(i) Stiffness Matrix is a Symmetric Matrix.

(ii) Stiffness Matrix is a banded Matrix.

(iii) If there are n number of nodes then global Stiffness Matrix is $n \times n$. provided element is a one dimensional and one degree of freedom at each node.

(iv) The Main diagonal Elements are always positive.

(~~v) rigid body~~

Equations to solve the one dimensional problems are.

1) Stiffness Matrix for each Element $= K = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

2) Assembly Stiffness Matrix $= K = K_1 + K_2 + \cdots = \begin{bmatrix} K_{11} & \cdots \\ \vdots & \ddots \\ \vdots & K_{nn} \end{bmatrix}$

3) FEM Equilibrium Equation.

$$F = KQ. = \begin{Bmatrix} F_1 \\ \vdots \\ F_n \end{Bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{Bmatrix} q_1 \\ \vdots \\ q_n \end{Bmatrix}$$

4) Stress at each Element.

$$\sigma_1 = \frac{E_1}{L_1}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

5) Reaction Forces at Node.

$$R = K_{11}q_1 + K_{12}q_2 + \cdots$$

$$\therefore \begin{Bmatrix} F_i \\ F_j \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

$$\boxed{F = K.u} \quad \text{or} \quad \boxed{F = KQ}$$

Where. $\boxed{K = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$ → Stiffness Matrix. For 1-D Bar Element

F → Force Matrix.

Q or u → displacement Matrix.

Global Stiffness Matrix.

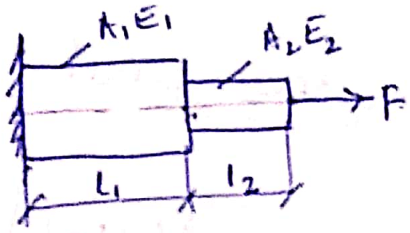
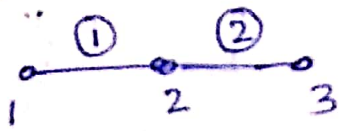


Figure shows the stepped shaft subjected to point load. The Global Stiffness Matrix written as follows.

FEA Model of the problem.



Element Connectivity table

Node	element
1	1-2 — 1
2	2-3 — 2
3	

For Element 1

$$K_1 = \frac{A_1E_1}{l_1}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \end{array}$$

For Element 2

$$K_2 = \frac{A_2E_2}{l_2}\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} 2 \\ 3 \end{array}$$

Global Stiffness Matrix is

$$K = K_1 + K_2$$

$$\therefore [K] = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{A_1E_1}{l_1} & -\frac{A_1E_1}{l_1} & 0 \\ -\frac{A_1E_1}{l_1} & \frac{A_1E_1}{l_1}+\frac{A_2E_2}{l_2} & -\frac{A_2E_2}{l_2} \\ 0 & -\frac{A_2E_2}{l_2} & \frac{A_2E_2}{l_2} \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

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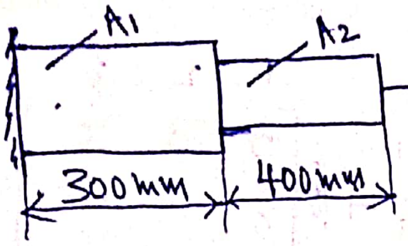
① Consider a stepped bar shown in Fig.



$A_1 = 2400\ mm^2$

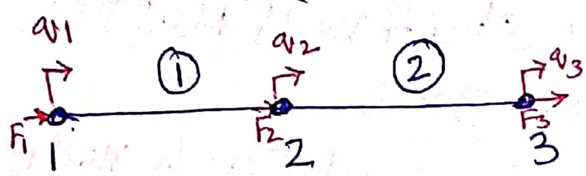
$A_2 = 600\ mm^2$

$E_1 = E_2 = 200\ GPa$

Determine Nodal displacement, Element stresses and reaction forces of axially loaded stepped bar as shown in Fig.

Solution:

Step 1: Finite element model of the given problem i.e. in terms of node and elements.



For Element ①

$A_1 = 2400\ mm^2$

$l_1 = 300\ mm$

$E_1 = 200 \times 10^3\ N/mm^2$

Element ②

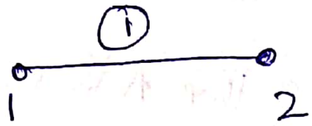
$A_2 = 600\ mm^2$

$l_2 = 400\ mm$

$E_2 = 200 \times 10^3\ N/mm^2$

Step 2: Stiffness matrix for each element.

For Element ①.

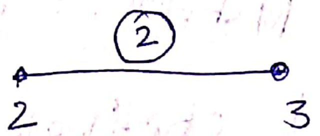


$$\therefore K_1 = \frac{A_1 E_1}{l_1} \begin{array}{cc} 1 & 2 \\ \end{array} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{array}{c} 1 \\ 2 \end{array}$$

$\therefore$ Stiffness Matrix for Element ① $K_1 = \frac{\overset{8}{\cancel{2}400} \times 200 \times 10^3}{\cancel{300}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$K_1 = 16 \times 10^5 \begin{array}{cc} 1 & 2 \\ \end{array} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{array}{c} 1 \\ 2 \end{array} = 10^5 \begin{array}{cc} 1 & 2 \\ \end{array} \begin{bmatrix} 16 & -16 \\ -16 & 16 \end{bmatrix} \begin{array}{c} 1 \\ 2 \end{array}$$

For Element ②



$$\therefore K_2 = \frac{A_2 E_2}{l_2} \begin{array}{cc} 2 & 3 \\ \end{array} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{array}{c} 2 \\ 3 \end{array}$$

$$\therefore K_2 = \frac{\overset{300}{\cancel{600}} \times \cancel{200} \times 10^3}{\cancel{2}\cancel{400}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad K_2 = 3 \times 10^5 \begin{array}{cc} 2 & 3 \\ \end{array} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{array}{c} 2 \\ 3 \end{array}$$

$$\therefore K_2 = 10^5 \begin{array}{cc} 2 & 3 \\ \end{array} \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \begin{array}{c} 2 \\ 3 \end{array}$$

Step 3: Global Stiffness Matrix or Assembly of Stiffness Matrix.

$$K = K_1 + K_2 = 10^5 \begin{array}{c|c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \hline \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 16 & -16 & 0 \\ -16 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} + 10^5 \begin{array}{c|c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \hline \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix} \end{array}$$

$$K = 10^5 \begin{array}{c|c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \hline \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 16 & -16 & 0 \\ -16 & 19 & -3 \\ 0 & -3 & 3 \end{bmatrix} \end{array}$$

Note: The Matrix size of the Stiffness Matrix for 1-D Element $n \times n$. i.e. 3×3.
because number of Nodes are 3 and each node having the dof for axial problem is one.
$\therefore$ 3 Nodes and 3 dof. i.e. 3×3

Step 4: Equating The Assembly Stiffness Matrix in FEM equilibrium equation we get

$$F = KQ.$$

where F = Force Matrix. $F = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$ forces at respective nodes.

K = Assembly Stiffness Matrix.

Q = Displacement Matrix. $Q = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$

Nodal displacement at each respective nodes.

$$\therefore \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = 10^5 \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 16 & -16 & 0 \\ -16 & 19 & -3 \\ 0 & -3 & 3 \end{bmatrix} \end{array} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

Step 5: Applying the Boundary Conditions

From figure boundary conditions $q_1 = 0$ i.e. Fixed end.

Using Gauss Elimination Methods of handling the boundary condition. By Eliminating the corresponding row and column with respect to zero boundary condition. i.e $q_1 = 0$. $q_2 = ?$ $q_3 = ?$

$f_1 = 0$ $\quad F_2 = 0$ $\quad F_3 = 200\,KN = 200 \times 10^3\,N$ Then

$$\begin{Bmatrix} 0 \\ 0 \\ 2\times10^5 \end{Bmatrix} = 10^5 \begin{bmatrix} 16 & -16 & 0 \\ -16 & 19 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ q_2 \\ q_3 \end{Bmatrix}$$

Matrix reduce to

Then

$$\begin{Bmatrix} 0 \\ 2\times10^5 \end{Bmatrix} = 10^5 \begin{bmatrix} 19 & -3 \\ -3 & 3 \end{bmatrix} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}$$

$$\therefore \quad 0 = (10^5 \times 19) q_2 - (10^5 \times 3) q_3$$

Step 6: $2\times10^5 = -(3\times10^5) q_2 + (3\times10^5) q_3$

By Solving the Simultaneous equations we get.

$q_2 = 0.125\,mm$ $\quad q_3 = 0.7916\,mm$

$\therefore$ Nodal displacements are. $\boxed{q_1 = 0 \quad q_2 = 0.125\,mm \quad q_3 = 0.7916\,mm}$

Step 7: Calculate the Stresses at each element.

$\cancel{f_1 = \frac{A_1 E_1}{L_1}}$ $\quad \sigma_1 = \frac{E_1}{L_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$

$$= \frac{200\times10^3}{300} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.125 \end{Bmatrix}$$

$$\therefore \boxed{\sigma_1 = 83.33\,N/mm^2}$$

at element 2

$$\sigma_2 = \frac{E_2}{l_2}\begin{bmatrix}1 & -1\end{bmatrix}\begin{Bmatrix}q_2\\ q_3\end{Bmatrix}$$

$$= \frac{200\times10^3}{400}\begin{bmatrix}1 & -1\end{bmatrix}\begin{Bmatrix}0.125\\ 0.7916\end{Bmatrix}$$

$$\boxed{\sigma_2 = 333.33\ N/mm^2}$$

Step 8: Reaction Force at Node 1.

$$R_1 = K_{11}q_1 + K_{12}q_2 + K_{13}q_3$$

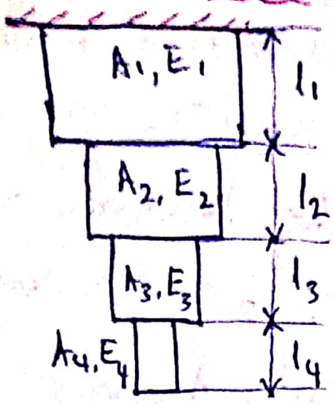
$$= \cancel{(16\times10^5)(0)} + (-16\times10^5)(0.125) + \cancel{(0)(0.7916)}$$

$$R_1 = -2\times10^5 = -200\times10^3 = -200KN$$

$$\therefore \boxed{R_1 = -200KN}$$

ASSIGNMENT PROBLEMS

Q① Figure shows stepped bar having cross section areas A_1, A_2, A_3 & A_4 and Young Modulus of E_1, E_2, E_3 & E_4, The length of the bars are l_1, l_2, l_3 and l_4. For given Stepped bar write the following.



(i) FEA Model.

(ii) Stiffness Matrix for each Element.

(iii) Global/Assembly Stiffness Matrix for the give bar.

Q.② Consider the bar shown in Fig. Determine

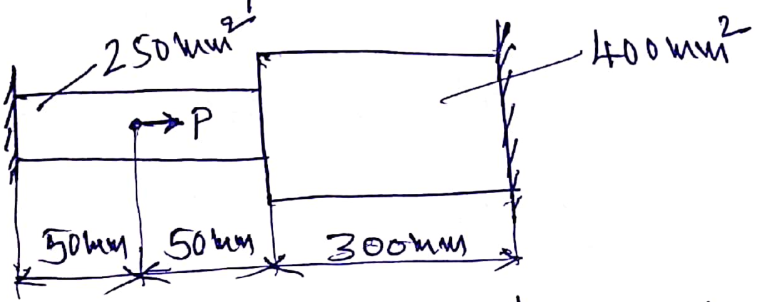
(i) Element Stiffness Matrix.

(ii) Nodal displacement.

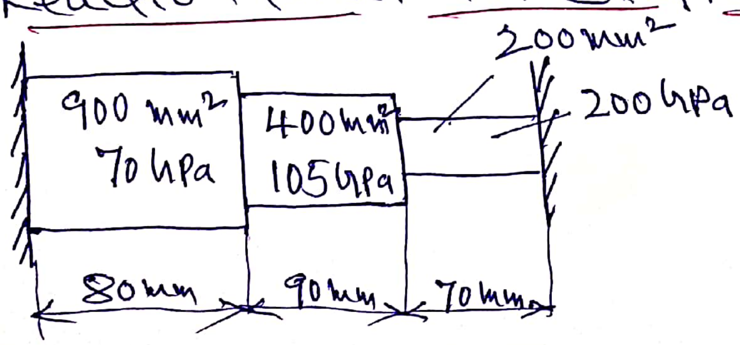
(iii) stresses at each element.

(iv) Reaction Forces.

take Youngs Modulus E = 200 GPa.



Q③. Find the Nodal displacement, Element stress and Reaction forces for stepped bar as shown in Fig.



Q(4) A stepped bar having different cross sectional area $A_1 = 275\, mm^2$, $A_2 = 125\, mm^2$ and stepped bar having the property of the Material E = 70 kN and bar is fixed at one End, The other End 150 KN load acting axially, The length of the stepped bar are $l_1 = 400\, mm$, $l_2 = 300\, mm$.

Find (i) FEA Model & Connectivity table.

(ii) Stiffness Matrix of each Element.

(iii) Global Stiffness Matrix.

(iv) Nodal displacement.

(v) Element Stress.

(vi) Reaction forces.

Penalty Approach for hadling boundary conditions

Penalty Approach Method is easy to implement in a computer program and retain its simplicity even when considering general boundary conditions. Specified displacement boundary condition will be discussed first. The Method will then be shown to apply to problems with Multi point constraints.

Consider That penalty approach presented herein is an approximate approach. The accuracy of the Solution, particularly the reaction forces, depends on the choice of C. If C is choosen large value of magnitude.

$$\boxed{C = Max|K_{ij}| \times 10^4}$$

Modify the Stiffness Matrix K by adding a large number C to each of diagonal Elements of K. Also modifying the global load vector F by adding Ca_1 to F_{P_1}, ---- to Ca_r to F_{P_i}. Solve $F = KQ$. For The displacement Q, where K and F are the Modified Stiffness and load Matrices.

Evaluate the reaction force at each Support from.

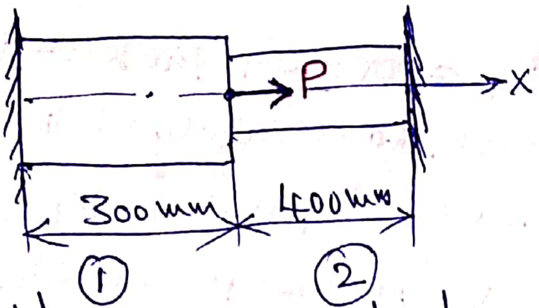
$$\boxed{R_{P_i} = -C(Q_{P_i} - a_i)} \quad i = 1, 2 - - - - r.$$

Q. Consider the bar shown in Fig. An axial load $P = 200 \times 10^3$ N is applied as shown. Using the penalty approach for handling boundary conditions, do the following.

(a) Determine the Nodal displacement

(b) Determine the Stress in each Material.

(c) Determine the reaction forces.

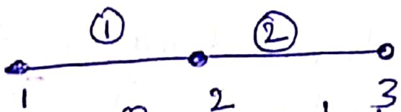


Aluminium. Steel

$A_1 = 2400\ mm^2$ $A_2 = 600\ mm^2$

$E_1 = 70 \times 10^9\ N/m^2$ $E_2 = 200 \times 10^9\ N/m^2$

Solution: FEA Model.



For Element 1

$$K_1 = \frac{A_1 E_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2400 \times 70 \times 10^3}{300}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \end{array} = 56 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_1 = 10^5 \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 56 & -56 \\ -56 & 56 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \end{array}$$

For Element 2

$$K_2 = \frac{A_2 E_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{600 \times 200 \times 10^3}{400}\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} 2 \\ 3 \end{array} = 0.3 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_2 = 10^6 \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 0.3 & -0.3 \\ -0.3 & 0.3 \end{bmatrix} \end{array}\begin{array}{c} 2 \\ 3 \end{array} =$$

Assembly Matrix. $K = K_1 + K_2 = 10^5 \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.3 \\ 0 & -0.3 & 0.3 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \end{array}$

FEM equilibrium equation.

$$F = KQ$$

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = 10^5 \begin{bmatrix} 0.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.3 \\ 0 & -0.3 & 0.3 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

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By applying boundary condition by penalty approach Method, Now Node 1 and Node 3 are fixed. Therefore a large Number C is added to First and Third diagonal Element.

$$C = [0.86 \times 10^6] \times 10^4$$

Thus the modified Stiffness Matrix is

$$K = 10^6 \begin{bmatrix} 8600.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 8600.30 \end{bmatrix}$$

The Finite Element equations are given by

$$\begin{Bmatrix} 0 \\ 200 \times 10^3 \\ 0 \end{Bmatrix} = 10^6 \begin{bmatrix} 8600.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 8600.30 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

by solving the simultaneous equations.

$$\boxed{q_1 = 15.1432 \times 10^{-6}\,\text{mm} \quad q_2 = 0.23257\,\text{mm} \quad q_3 = 8.1127 \times 10^{-6}\,\text{mm}}$$

Nodal displacements are q_1, q_2 and q_3 as above.

Element Stresses.

$$\sigma_1 = \frac{E_1}{l_1}[-1 \;\; 1]\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}; \quad \sigma_1 = \frac{70 \times 10^3}{300}[-1 \;\; 1]\begin{Bmatrix} 15.1432 \times 10^{-6} \\ 0.23257 \end{Bmatrix} = \boxed{54.27\ \text{N/mm}^2}$$

$$\sigma_2 = \frac{E_2}{l_2}[-1 \;\; 1]\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} \quad \sigma_2 = \frac{200 \times 10^3}{400}[-1 \;\; 1]\begin{Bmatrix} 0.23257 \\ 8.1127 \times 10^{-6} \end{Bmatrix} = \boxed{-116.29\ \text{N/mm}^2}$$

Reaction Forces

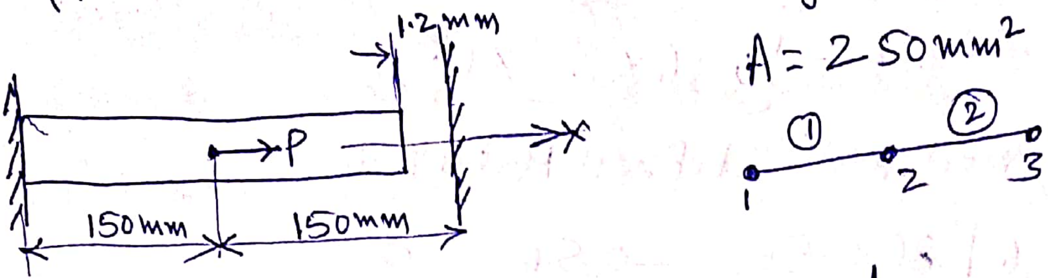
at Node 1: $R_1 = -Cq_1 = -[0.86 \times 10^{10}][15.1432 \times 10^{-6}] = -130.23 \times 10^3$

$$\boxed{R_1 = -130.23\,\text{KN}}$$

at Node 3: $R_3 = -Cq_3 = -[0.86 \times 10^{10}][8.1127 \times 10^{-6}] = -69.77 \times 10^3$

$$\therefore \boxed{R_3 = -69.27\,\text{KN}}$$

In Fig. a load $P = 60\times10^3$ N is applied as shown. Determine the displacement field, stress and support reaction in the body; take $E = 20\times10^3$ N/mm²



$\frac{AE}{L} = \frac{250\times20\times10^3}{150}$

Solution! The F.E equation bar element.

$$K = \frac{AE}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$K = \frac{AE}{L}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\end{bmatrix} = \begin{bmatrix}F_1\\ F_2\\ F_3\end{bmatrix}$$

$q_1 = 0 \quad q_2 = ? \quad q_3 = 1.2\text{ mm} \quad P = 60\times10^3$ N.

$$\frac{AE}{L}\begin{bmatrix}\cancel{1} & \cancel{-1} & \cancel{0}\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}\cancel{0}\\ q_2\\ \Delta\end{bmatrix} = \begin{bmatrix}\cancel{F_1}\\ 60\times10^3\\ F_3\end{bmatrix}$$

$$\frac{AE}{L}\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}q_2\\ \Delta\end{bmatrix} = \begin{bmatrix}60\times10^3\\ F_3\end{bmatrix}$$

$$\frac{AE}{L}[2\ \ -1]\begin{Bmatrix}q_2\\ \Delta\end{Bmatrix} = 60\times10^3$$

$$\frac{250\times20\times10^3}{150}[2\ \ -1]\begin{bmatrix}q_2\\ 1.2\end{bmatrix} = 60\times10^3$$

$$33.33\times10^3[2\ \ -1]\begin{Bmatrix}q_2\\ 1.2\end{Bmatrix} = 60\times10^3$$

$$33.33\times10^3(2)(q_2) - 33.33\times10^3(1.2) = 60\times10^3$$

$$33.33\times10^3\times2q_2 = 60\times10^3 + 33.33\times10^3\times1.2$$

$$\therefore q_2 = \frac{\cancel{10^3}(60+39.99)}{33.33\times\cancel{10^3}\times2} = 1.5\text{ mm}$$

$$\therefore \boxed{q_2 = 1.5\text{ mm}}$$

Reaction at Node 1: $R_1 = \cancel{K_{11}q_1}^{0} + K_{12}q_2 + \cancel{K_{13}q_3}^{0}$

$$= -1\left(\frac{AE}{L}\right)(q_2) = -(1)\left[\frac{250\times20\times10^3}{150}\right][1.5]$$

$$\boxed{R_1 = -50\times10^3\text{ N}} = -50\text{ kN}$$

Reaction at Node 3:

$$R_3 = \cancel{K_{31}q_1}^{0} + K_{32}q_2 + K_{33}q_3 = -(1)(33.33)(1.5) + 33.33(1.2)$$

$$= -49.995 + 39.996 = -10\times10^3\text{ N}$$

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Stress at element ① & ②

$$\sigma_1 = \frac{E}{l_1}(-1 \;\; 1)\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{200\times10^3}{150}(-1 \;\; 1)\begin{Bmatrix} 0 \\ 1.5 \end{Bmatrix} = -200 \ N/mm^2$$

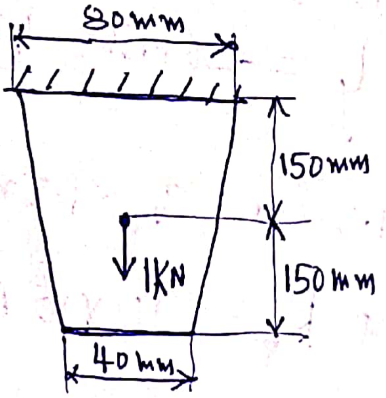
$$\sigma_2 = \frac{E}{l_2}(-1 \;\; 1)\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = 133.33(-1 \;\; 1)\begin{Bmatrix} 1.5 \\ 1.2 \end{Bmatrix} = -200+160 = -40 \ N/mm^2$$

Nodal Displacement; $q_1 = 0$; $q_2 = 1.5 \, mm$; $q_3 = 1.2 \, mm$

Reaction Forces; $R_1 = -50\times10^3 N$ & $R_3 = -10\times10^3 N$

TAPER BAR PROBLEM

For a taper bar shown in Fig. The Bar is subjected to a Point load $P = 1\,KN$ at its mid point. The density of bar is $7800\,kg/m^3$, Thickness of taper bar is 10 mm. Find the following



(i) Model the taper bar into Two Elements

(ii) Each Element Stiffness Matrix.

(iii) Global Stiffness Matrix.

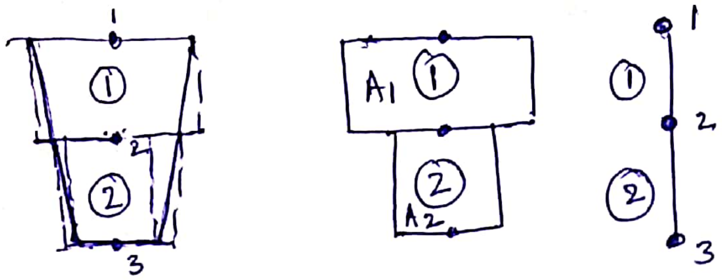
(iv) Nodal displacements

(v) Evaluate stress at each element.

(vi) Determine the Reaction forces.

Take $E = 200\,GPa$

Solution: (i) Model the taper bar into Two Elements



FEA Model.

$E = 200 \times 10^3\, N/mm^2$

at Node 1 c/s Area $= w \times t = 80 \times 10 = 800\,mm^2$

at Node 2 c/s Area $= \left(\frac{80+40}{2}\right) \times 10 = 60 \times 10 = 600\,mm^2$

at Node 3 c/s Area $= w \times t = 40 \times 10 = 400\,mm^2$.

$\therefore$ c/s Area for Element ① $= \frac{800+600}{2} = 700\,mm^2 = A_1$

c/s Area for Element ② $= \frac{600+400}{2} = 500\,mm^2 = A_2$

Young's Modulus $= E_1 = E_2 = 200 \times 10^3\, N/mm^2$

length of the Elements $l_1 = 150\,mm$; $l_2 = 150\,mm$

(ii) Element Stiffness Matrix For Element ① & Element ②

$$K_1 = \frac{A_1 E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{700 \times 200 \times 10^3}{150} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array} = 9.33 \times 10^5 \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array}$$

$$K_2 = \frac{A_2 E_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{500 \times 200 \times 10^3}{150} \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} 2 \\ 3 \end{array} = 6.66 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(iii) Global stiffness matrix

$$K = K_1 + K_2 = 10^5 \begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{bmatrix} 9.33 & -9.33 \\ -9.33 & 9.33 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} \end{array} + 10^5 \begin{array}{c} \begin{matrix} 2 & 3 \end{matrix} \\ \begin{bmatrix} 6.66 & -6.66 \\ -6.66 & 6.66 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix} \end{array}$$

$$K = 10^5 \begin{bmatrix} 9.33 & -9.33 & 0 \\ -9.33 & 15.99 & -6.66 \\ 0 & -6.66 & 6.66 \end{bmatrix}$$

$g = 9.81\ m/s^2$

$= 0.7 \times 10^{-4} \times 0.15 \times 7800 \times 9.81$

$= 0.7 \times 0.15 \times 7800 \times 9.81$

$= \cancel{8.062}\ 80.262\ N$

The forces at Node 1. $F_1 = V\rho g = A_1 l_1 \rho g = 700 \times 150 \times 7800 \times 9.81 \times 10^{-3} \times 10^{-9}$

W_1

$W_1 = 7 \times 10^2 \times 1.5 \times 10^2 \times 7.8 \times 10^3 \times 9.81 \times 10^{-9} \times 10^{-3}$

$= 7 \times 1.5 \times 7.8 \times 9.81 \times 10^{-12} \times 10^{7}$

$= 803.439 \times 10^7 \times 10^{-9} \times \cancel{10^{-3}}$

$= 8.034\ N$

element ② $W_2 = A_2 l_2 \rho g = 500 \times 150 \times 7800 \times 10^{-9} \times 9.81$

$= 5.739\ N$

$P = 1\ kN$. (Point load) at node 2

The Nodal force at Node 2.

$F_1 = \dfrac{\text{Body force of element 1}}{2} = \dfrac{8.034}{2} = 4.017\ N$

at Node 2, $F_2 = \dfrac{\text{Body force of element 1}}{2} + \dfrac{\text{B.F. element ②}}{2} + P$

$F_2 = 4.017 + \dfrac{5.739}{2} + 1000 = 1006.887\ N$

at Node 3, $F_3 = \dfrac{\text{Body force of element ②}}{2} = \dfrac{5.739}{2} = 2.870\ N$

F_3

$$\therefore\ F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 4.017 \\ 1006.88 \\ 2.870 \end{bmatrix}$$

$$10^5 \begin{bmatrix} 9.33 & -9.33 & 0 \\ -9.33 & 15.99 & -6.66 \\ 0 & -6.66 & 6.66 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 4.017 \\ 1006.88 \\ 2.870 \end{bmatrix}$$

The Boundary Condition. $q_1 = 0$

$$10^5 \begin{bmatrix} 15.99 & -6.66 \\ -6.66 & 6.66 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1006.88 \\ 2.870 \end{bmatrix}$$

$$10^5 \times 15.99 \times q_2 - 6.66 \times \cancel{10^5} q_3 = 1006.88$$

$$-6.66 \times 10^5 q_2 + \cancel{6.66} \times 10^5 q_3 = 2.870$$

$$10^5 \times 9.33\, q_2 = 1009.750$$

$$\therefore q_2 = \frac{1009.750}{9.33} \times 10^{-5} = 108.226 \times 10^{-5} \text{ mm}$$

$$-6.66 \times \cancel{10^5} \times 108.226 \times \cancel{10^{-5}} + 6.66 \times 10^5 q_3 = 2.870$$

$$723.655\, q_3 = 2.870 \qquad \therefore \cancel{q_3} = \frac{2.870}{723.655} \cancel{= 3.96}$$

$$666 \times 10^5 q_3 = -723.655$$

$$q_3 = -\frac{723.655}{6.66} \times 10^{-5} = 108.656 \times 10^{-5} \text{ mm}$$

Stress at element ① $\sigma_1 = \frac{E}{L_1}(-1\ 1)\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \frac{2\times 10^5}{150}(-1\ 1)\begin{Bmatrix} 0 \\ 108.226\times 10^{-5} \end{Bmatrix}$

$$\sigma_1 = 0.0133 \times 10^5 (-1\ 1)\begin{Bmatrix} 0 \\ 108.226 \times 10^{-5} \end{Bmatrix}$$

$$\sigma_1 = -0.0133 \cancel{\times 10^5 \times 0} + 0.0133 \times 10^5 \times 108.226 \times 10^{-5} = 1.44 \text{ N/mm}^2$$

(ii) $\sigma_2 = \frac{E}{L_2}(-1\ 1)\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \frac{2\times 10^5}{150}(-1\ 1)\begin{Bmatrix} 108.226\times 10^{-5} \\ 108.656\times 10^{-5} \end{Bmatrix}$

$$\sigma_2 = -0.0133 \times \cancel{10^5} \times 108.226 \times \cancel{10^{-5}} + 0.0133 \times \cancel{10^5} \times 108.656 \times \cancel{10^{-5}} =$$

$$= -1.4394 + 1.445 = 5.6 \times 10^{-3} \text{ N/mm}^2$$

Two dimensional Analysis

Dr. HSS

Session:

Two dimensional Elements

Learning out comes:

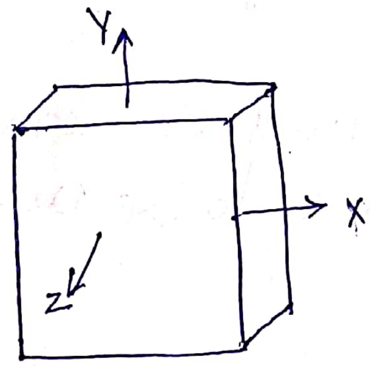
1. Gain knowledge about types of two dimensional problems.
2. Understand 2 D. Finite Element formulation.
3. Solve simple 2D engineering problems.

Two dimensional Analysis.

* Analysis of plane bodies.
  - Bodies which are flat and have constant thickness
* Analysis of two-dimensional problems
  - plane stress
  - plane strain and
  - Axisymmetric problems.

plane stress

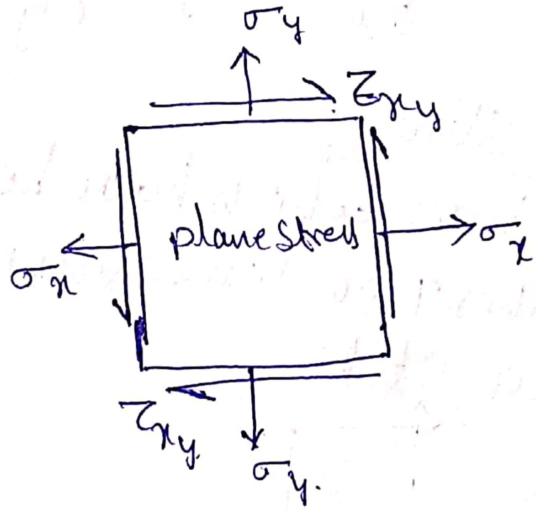
* Structures which are thin (small thickness) in comparison to other two dimensions.
* Subjected to in-plane loads.



plane stress
(thin body)

$\sigma_z = 0$

$\tau_z = 0$



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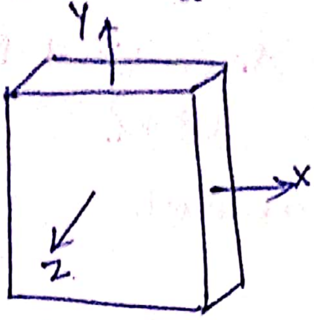
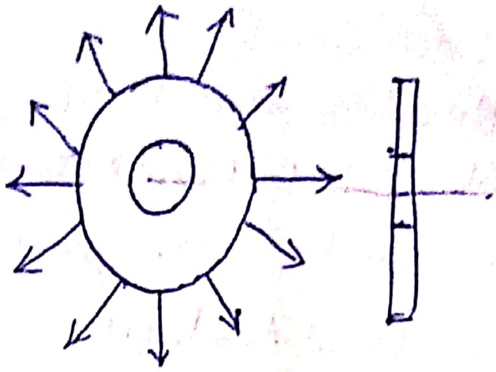
* Since the Thickness is small stresses along thickness can be neglected
ie $\sigma_z = \tau_{xz} = \tau_{yz} = 0$



plane stress.
$\sigma_z = 0, \tau_z = 0$



Thin disk under stress

Eg: Thin disk, rotating impeller wheels, Thin plate under tension.

Stress-Strain Relations

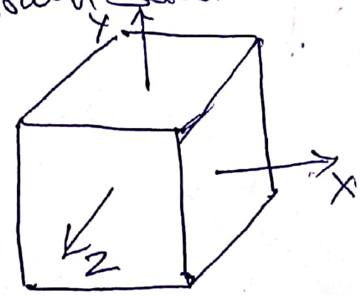
- For linear, elastic, isotropic Material

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \left(\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} - \{\epsilon\}^0 \right)$$

$$\epsilon_x = \frac{\partial u}{\partial x} \quad \epsilon_y = \frac{\partial v}{\partial y} \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Plane Strain:

Structures which have large thickness in comparison to other two dimensions are said to be under plane strain state.

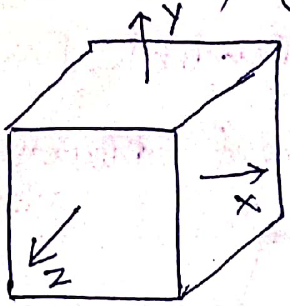


Thick body

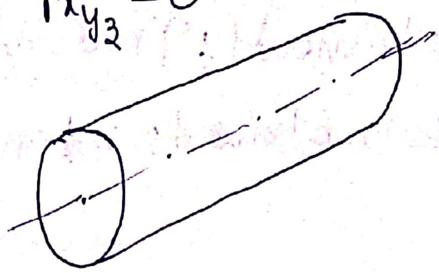
$\epsilon_z = 0 \quad \sigma_z = \nu(\sigma_x + \sigma_y)$

Scanned with CamScanner

- Since the length of the member is long, strain along thickness (z-direction) can be assumed to be zero. i.e., $\varepsilon_z = \varepsilon_{xz} = \gamma_{xyz} = 0$



Plane strain (Thick body)



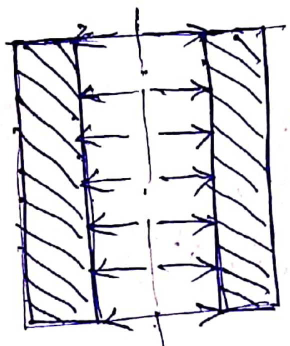
A long prismatic shaft subjected to torsion.

$\tau_z = 0$; $\sigma_z = \nu(\sigma_x + \sigma_y)$

Ex: Torsion of long uniform shaft, long cylinders subjected to internal pressure (tunnel), Retaining wall of a dam.

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\left(\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} - \{\varepsilon\}^0\right)$$

Axisymmetric problems.



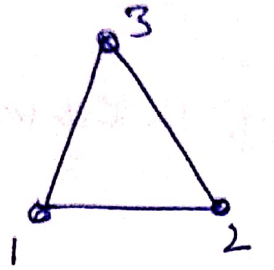
Tick cylinder under Internal pressure.

* Class of 3D problem (solids of revolution) where geometry of structure, material properties and boundary conditions (load and support) are axisymmetric.
* Field variable can be assumed not to vary along circumferential direction
* It becomes sufficient to determine variation of field variable in any one plane.
* 3D reduce to 2D problems

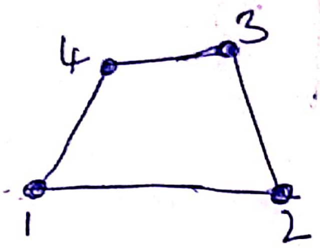
Ex: pressure vessels.

Two dimensional Elements

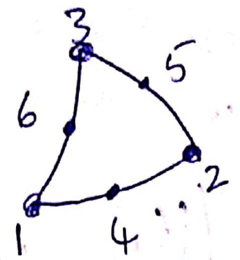
* Plane elements formulated to represent two-dimensional geometry.
* Common elements: Triangular and Quadrilateral in shape
* Typical 2D-Elements: Linear Element & Quadratic Element



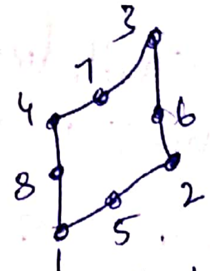
3-Noded Triangular Element



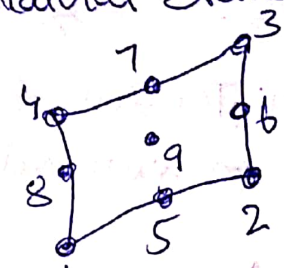
4-Noded Quadrilateral Element.



6-Noded Triangular Element.



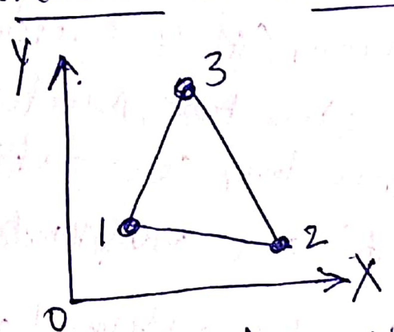
8-Noded Quadrilateral Element (Serendipity)



9-Noded Quadrilateral element (Lagrangian)

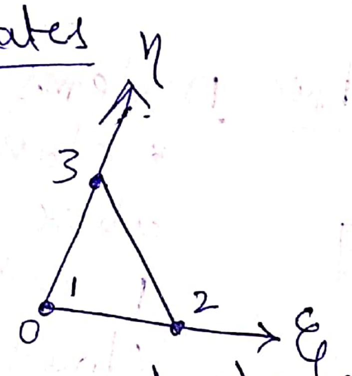
Co-ordinate System

Global and local co-ordinates



Derived element

Global Co-ordinates.

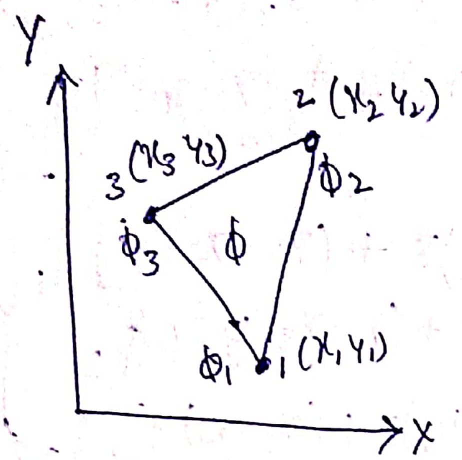


Parent element

local-co-ordinates

DS-185

Derivation of shape function for CST element (3-Noded Triangular Element) in Global co-ordinates.



- Consider a general 3-noded triangular Element.
- Field Variable. $\phi(x,y) = a_0 + a_1 x + a_2 y$ — (1)
- Nodal conditions. $\phi(x_1, y_1) = \phi_1$

$\phi(x_2\, y_2) = \phi_2$

$\phi(x_3\, y_3) = \phi_3$

- Substituting, we get.

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

solving for polynomial co-efficent, we get

$$a_0 = \frac{1}{2A}\left[\phi_1(x_2y_3 - x_3y_2) + \phi_2(x_3y_1 - x_1y_3) + \phi_3(x_1y_2 - x_2y_1)\right]$$

$$a_1 = \frac{1}{2A}\left[\phi_1(y_2 - y_3) + \phi_2(y_3 - y_1) + \phi_3(y_1 - y_2)\right]$$

$$a_2 = \frac{1}{2A}\left[\phi_1(x_3 - x_2) + \phi_2(x_1 - x_3) + \phi_3(x_2 - x_1)\right]$$

where A is the area of triangle given by

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

Substituting for co-efficients a_0, a_1 and a_2

$$\phi(x, y) = a_0 + a_1 x + a_2 y$$

$$\phi(x,y) = \frac{1}{2A}\left\{\begin{array}{l}[\phi_1(x_2y_3 - x_3y_2) + \phi_2(x_3y_1 - x_1y_3) + \phi_3(x_1y_2 - x_2y_1)] \\ + [\phi_1(y_2 - y_3) + \phi_2(y_3 - y_1) + \phi_3(y_1 - y_2)]x \\ + [\phi_1(x_3 - x_2) + \phi_2(x_1 - x_3) + \phi_3(x_2 - x_1)]y\end{array}\right\} \quad \text{--- (1)}$$

Simplifying, we get

$$\phi(x,y) = \frac{1}{2A}\left\{\begin{array}{l}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]\phi_1 \\ +[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y]\phi_2 \\ +[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y]\phi_3\end{array}\right\} \quad \text{--- (2)}$$

Field variable interpolated in terms of nodal values.

$$\phi(x,y) = N_1\phi_1 + N_2\phi_2 + N_3\phi_3 \quad \text{--- (3)}$$

N_1, N_2 and N_3 are called Shape functions.

Comparing EQ (2) and EQ (3)

$$N_1(x,y) = \frac{1}{2A}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$

$$N_2(x,y) = \frac{1}{2A}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y]$$

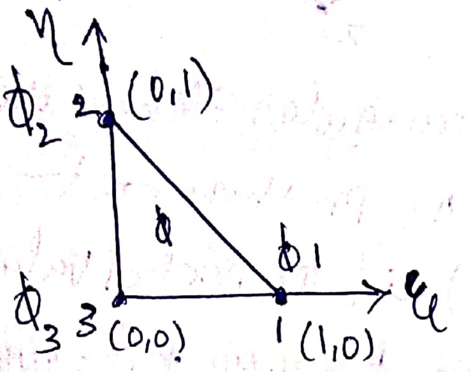
$$N_3(x,y) = \frac{1}{2A}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y]$$

Dr.HSS

Derivation of Shape functions for CST Element (3-Noded Triangular Element) in Natural co-ordinates.

Consider a general 3-Noded triangular element,

field variable $\phi(\xi, \eta) = a_0 + a_1 \xi + a_2 \eta$ — (1)



nodal conditions.

At Node 1, $\phi(1,0) = \phi_1$

At Node 2, $\phi(0,1) = \phi_2$

At Node 3, $\phi(0,0) = \phi_3$

Substituting, we get

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

Solving for polynomial Co-efficent, we get

$a_0 = \phi_3 \quad a_1 = \phi_1 - \phi_3 \quad a_2 = \phi_2 - \phi_3$

Substituting for Co-efficent a_0, a_1 and a_2

$$\phi(\xi, \eta) = a_0 + a_1 \xi + a_2 \eta$$

$$\phi(\xi, \eta) = \phi_3 + (\phi_1 - \phi_3)\xi + (\phi_2 - \phi_3)\eta \quad \text{— (2)}$$

Field variable interpolated in terms of nodal values

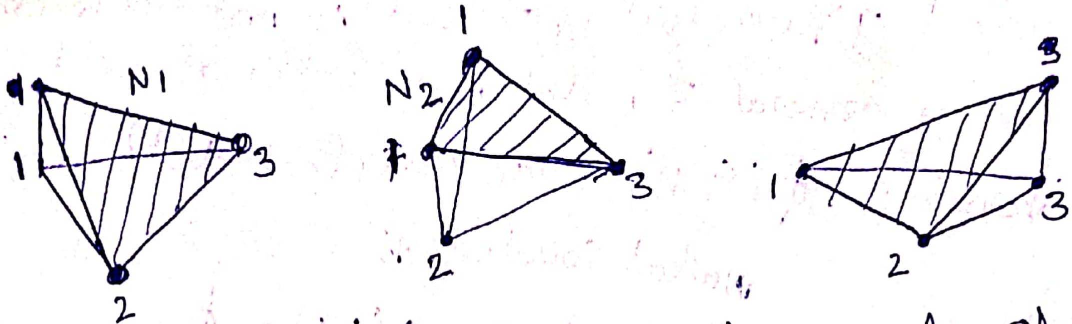
$$\phi(\xi, \eta) = N_1 \phi_1 + N_2 \phi_2 + N_3 \phi_3 \quad \text{— (3)}$$

N_1, N_2 and N_3 are called Shape function

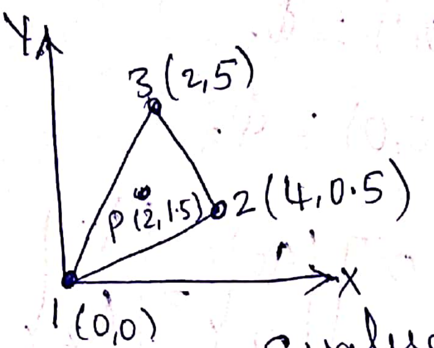
Comparing eq (2) and eq (3).

$$N_1 = \xi \qquad N_2 = \eta \qquad N_3 = 1 - \xi - \eta$$

Variation of Shape functions:



① Evaluate the shape functions for triangular element shown. Also evaluate the pressure at point P (2, 1.5) if the nodal values are $\phi_1 = 40 MPa$, $\phi_2 = 34 MPa$ & $\phi_3 = 46 MPa$



Evaluation of shape functions.

$(x_1, y_1) = (0,0)$ $(x_2, y_2) = (4, 0.5)$ $(x_3, y_3) = (2, 5)$

Method 1:

$$N_1(x,y) = \frac{1}{2A}\left[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y\right]$$

$$N_2(x,y) = \frac{1}{2A}\left[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y\right]$$

$$N_3(x,y) = \frac{1}{2A}\left[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y\right]$$

Substituting, we get $2A = 19$ and

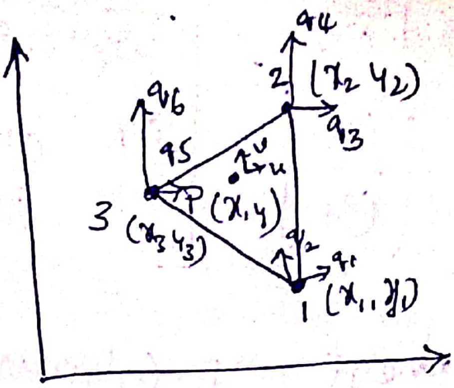
$$N_1 = \frac{19 - 4.5x - 2y}{19}$$

$$N_2 = \frac{5x - 2y}{19}$$

$$N_3 = \frac{-0.5x + 4y}{19}$$

Derivation of Strain-displacement Matrix of CST Element.

Geometry.



$x = N_1x_1 + N_2x_2 + N_3x_3$

$= \xi x_1 + \eta x_2 + (1-\xi-\eta)x_3$

$= \xi(x_1 - x_3) + \eta(x_2 - x_3) + x_3$

$x = \xi x_{13} + \eta_{23} + x_3$

$y = N_1y_1 + N_2y_2 + N_3y_3$

$y = \xi y_{13} + \eta y_{23} + y_3$

Displacement.

$u = N_1q_1 + N_2q_3 + N_3q_5$

$= \xi q_1 + \eta q_3 + (1-\xi-\eta)q_5$.

$u = \xi q_{15} + \eta q_{35} + q_5$

$v = \xi q_{26} + \eta q_{26} + q_6$

Strain-displacement relation

$\epsilon_x = \frac{\partial u}{\partial x}$ $\epsilon_y = \frac{\partial v}{\partial y}$ $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$

Displacement u, v and Geometry x, y: function of ξ, η.

partial derivatives of u and v to be taken w.r.t x and y

Considering partial derivatives of u: using chain rule.

$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \xi}$

$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \eta}$

In Matrix form.

$$\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = J \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix}$$

where Jacobian $J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}$

$x = \xi x_{13} + \eta x_{23} + x_3$

$y = \xi y_{13} + \eta y_{23} + y_3$

Now $$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix}$$

Now $J^{-1} = \frac{1}{|J|} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix}$ and $|J| = x_{13} y_{23} - x_{23} y_{13}$

$u = \xi q_{15} + \eta q_{35} + q_5$

Hence
$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} q_{15} \\ q_{35} \end{Bmatrix}$$

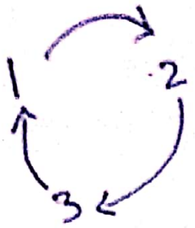
|||ly
$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} q_{26} \\ q_{46} \end{Bmatrix}$$

Strain

$$\epsilon = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\[2mm] \dfrac{\partial v}{\partial y} \\[2mm] \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\[2mm] \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} q_{15} \\ q_{35} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\[2mm] \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} q_{26} \\ q_{46} \end{Bmatrix}$$



$$= \frac{1}{|J|} \begin{Bmatrix} y_{23}(q_1 - q_5) - y_{13}(q_3 - q_5) \\ -x_{23}(q_2 - q_6) + x_{13}(q_4 - q_6) \\ -x_{23} q_{15} + x_{13} q_{35} + y_{23} q_{26} - y_{13} q_{46} \end{Bmatrix}$$

$$\epsilon = \frac{1}{|J|} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{13} & x_{21} & y_{12} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

$$\epsilon = B q$$

where

$$B = \frac{1}{|J|} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

is called Strain-displacement Matrix