

Module -5:Determinants:

- * In linear algebra, the determinant is a scalar value that can be computed from the elements of a square matrix. and encodes certain properties of the linear transformation described by the matrix.
- * The determinant of a matrix A is denoted by $\det(A)$, $\det A$ or $|A|$.

Properties of determinants:

- 1) The determinant of the identity matrix is 1.

$$\boxed{\det I = 1} \quad \left| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right|_{2 \times 2} = 1. \text{ and}$$

$$\left| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right|_{3 \times 3} = 1.$$

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- ② The determinant changes sign when two rows or columns are exchanged, then sign of determinant changes.

$$\text{Row Exchange} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cd.$$

$$= - \begin{vmatrix} c & d \\ a & b \end{vmatrix}.$$

The determinant of every permutation matrix is $\det P = \pm 1$

- ③ If two rows of A are equal, then $\det A = 0$.

$$\text{equal rows} = \begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0.$$

- ④ If A has a row of zeros, then $\det A = 0$

$$\text{Zero row} = \begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0.$$

- ⑤ The value of determinant remains unchanged if its rows and columns are interchanged

Transpose rule $|\mathbf{A}| = |\mathbf{A}^T|$

$$\text{eg: } |\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

$$|\mathbf{A}^T| = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - cb.$$

⑥ If A is singular, then $\det A = 0$.

Singular matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det A \mid \begin{array}{cc|c} a & b \\ c & d \end{array} \mid = ad - bc = 0.$$

No singular matrix $\det A \neq 0$.

⑦ The determinant of AB is the product of $\det A$ times $\det B$.

Product rule $|\mathbf{A}| |\mathbf{B}| = |\mathbf{A} \mathbf{B}|$.

⑧ Det is linear in each row separately.

a) Subtracting a multiple of one row from another row leaves the determinant unchanged.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} a & b \\ c-k a & d-k b \end{bmatrix}$$

$$|A| = \underline{\underline{ad - bc}}$$

$$|B| = (a)(d-kb) - (b)(c-ka)$$

$$= ad - akb - bc + akb$$

$$= \underline{\underline{ad - bc}}$$

Co-factor:

The determinant of an $n \times n$ matrix

$A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{ij} \det A_{ij}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \dots, a_{1n}$ are from the first row A .

i.e

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \dots + (-1)^{1+n} a_{1n} \det A_{1n}$$

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a 3×3 matrix.

$$\det A = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\det A = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} - a_{12}a_{31}a_{23} \\ + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}.$$

This formula is called cofactor expansion across the first row A .

$$\text{i.e } C_{ij} = (-1)^{i+j} \det A_{ij}.$$

where $i = \text{row}$

$j = \text{column}$.

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

$$a_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$a_{21} = (-1)^{2+1} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Problems: (Problem Set 4-2 → Gilbert Strang)

1) If a 4 by 4 matrix has $\det A = \frac{1}{2}$

Find i) $\det(2A)$

ii) $\det(-A)$

iii) $\det(A^2)$

iv) $\det(A^{-1})$.

Solutions:

i) $\det(2A) = ?$. 4x4 matrix.

$$2^4 \det A.$$

$$= 16 \cdot \frac{1}{2} = \underline{\underline{\underline{\underline{8}}}}$$

ii) $\det(-A) = (-1)^4 \det A$

$$= 1 \cdot \frac{1}{2} = \underline{\underline{\underline{\underline{\frac{1}{2}}}}}$$

iii) $\det(A^2) = \det A \cdot \det A$

$$= \frac{1}{2} \cdot \frac{1}{2} = \underline{\underline{\underline{\underline{\frac{1}{4}}}}}$$

iv) $\det(A^{-1}) = \frac{1}{\det A} = \frac{1}{\frac{1}{2}} = \underline{\underline{\underline{\underline{2}}}}$

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2) If a 3 by 3 matrix has $\det A = -1$.

find) $\det \left(\frac{1}{2} A \right)$

2) $\det (-A)$

3) $\det (A^2)$

4) $\det (A^{-1})$.

Solutions:

1) $\det \left(\frac{1}{2} A \right) =$

$$= \left(\frac{1}{2} \right)^3 \det A$$

$$= \frac{1}{8} (-1) = \underline{\underline{-\frac{1}{8}}}.$$

2) $\det (-A)$.

$$= (-1)^3 \det A = (-1)(-1).$$

$$= \underline{\underline{1}}$$

3) $\det (A^2)$.

$$= \det A \cdot \det A = (-1)(-1) = \underline{\underline{1}}$$

4) $\det (A^{-1})$

$$= \frac{1}{\det A} = \frac{1}{-1} = \underline{\underline{-1}}$$

③ Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 5 & -2 \\ 1 & -3 & 4 \end{bmatrix}$ & $B = \begin{bmatrix} 3 & 2 & 1 \\ -4 & 5 & -1 \\ 2 & -3 & 4 \end{bmatrix}$

Find $\det(A)$ & $\det(B)$.

Solution:

$$\begin{aligned}\det(A) &= 2(20 - 6) - 1(0 + 2) + 1(0 - 5) \\ &= 2(14) - 1(2) + 1(-5) \\ &= 28 - 2 - 5 \\ &= \underline{\underline{21}}\end{aligned}$$

$$\begin{aligned}\det(B) &= 3(20 - 3) - 2(-16 + 2) + 1(12 - 10) \\ &= 3(17) - 2(-14) + 1(2) \\ &= 51 + 28 + 2 \\ &= \underline{\underline{81}}\end{aligned}$$

Cramer's rule:

It is used to solve the linear equations.

Eg:

$$\begin{aligned} x+2y+4z &= 1 \\ 2x+3y-z &= 3 \\ x-3z &= 2 \end{aligned}$$

Find x, y, z

using cramer's rule

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 1 & 0 & -3 \end{bmatrix}, D = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$\begin{aligned} |A| &= 1(9-0) - 2(-6+1) + 4(0-3) \\ &= -9 - 2(-5) + 4(-3) \end{aligned}$$

$$|A| = -9 + 10 - 12$$

$$|A| = \underline{\underline{-11}}$$

$$x = \frac{\begin{vmatrix} 1 & 2 & 4 \\ 3 & 3 & -1 \\ 2 & 0 & -3 \end{vmatrix}}{-11} = \frac{1(-9) - 2(-9+2) + 4(0-6)}{-11}$$

$$x = \frac{-9 - 14 + 24}{-11} = \frac{19}{11}$$

$$y = \frac{\begin{vmatrix} 1 & 1 & 4 \\ 2 & 3 & -1 \\ 1 & 2 & -3 \end{vmatrix}}{-11} = \frac{1(-9+2) - 1(-6+1) + 4(0-3)}{-11}$$

$$\textcircled{6} \quad y = \frac{-7 + 5 + 4}{-11} = \frac{2}{-11} = \frac{-2}{11}$$

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$$Z = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 1 & 0 & 2 \end{vmatrix}$$

-11

$$\Rightarrow \frac{1(6) - 2(u-3) + 1(t-3)}{-11}$$

$$Z \Rightarrow \frac{6 - 2 - 3}{-11} = \frac{1}{-11} = \underline{\underline{\frac{-1}{11}}}$$

$$x = \frac{19}{11}, y = -\frac{2}{11}, z = \underline{\underline{\frac{-1}{11}}}$$

Eigen values and Eigen vectors:

Let A be a square matrix of order n over the field F . An element λ in field F is called an eigen value of A if $|A - \lambda I| = 0$, where, I is the unit matrix of order n .

Eg: 3×3 matrix.

$$\text{Let } A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Characteristic matrix.

$$[A - \lambda I] = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

Characteristic polynomial:

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = -\lambda^3 + 7\lambda^2 - 11\lambda + 5$$

Characteristic Equation:

$$\begin{aligned} |A - \lambda I| &= 0 \\ &= -\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0 \\ &= \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0. \end{aligned}$$

$\lambda = 1, 1, 5$, \rightarrow Eigen values.

Characteristic values or Eigen values.

Problems①:

1) Find the Eigen values of matrix,

$$A \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$$

Solution:

1) characteristic matrix of A

$$[A - \lambda I] = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{bmatrix}$$

2) characteristic polynomial of A .

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} = (3-\lambda)(-\lambda) + 2 \\ &= -3\lambda + \lambda^2 + 2 \\ &= \lambda^2 - 3\lambda + 2. \end{aligned}$$

3) Characteristic equation:

$$|A - \lambda I| = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda^2 - 2\lambda - 1\lambda + 2 = 0$$

$$\lambda(\lambda-2) - (\lambda-2) = 0$$

$$(\lambda-1)(\lambda-2) = 0$$

$$\lambda = 1, \lambda = 2.$$

Eigen values of A are 1, 2.

Problem ②:-

Find the eigen values of matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$

Solution:

1) characteristic matrix of A

$$(A - \lambda I) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8-\lambda \end{bmatrix}$$

② Characteristic polynomial of A:

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8-\lambda \end{vmatrix}$$

$$= (-\lambda) [(-\lambda)(8-\lambda) + 17] - 1[0 - 4] + 0.$$

$$= (-\lambda) [-8\lambda^2 + \lambda^2 + 17] + 4.$$

$$= +8\lambda^2 - \lambda^3 - 17\lambda + 4.$$

$$= -\lambda^3 + 8\lambda^2 - 17\lambda + 4$$

3) characteristic equation of A

$$|A - \lambda I| = 0$$

$$= -\lambda^3 + 8\lambda^2 - 17\lambda + 4 = 0.$$

$$= \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0.$$

$$\lambda = 4, 2 \pm \sqrt{3}$$

Hence the eigen values of A are $4, 2 \pm \sqrt{3}$

Eigen Vectors:

Let λ be an eigen value of n square matrix A , then a non-zero matrix X of the order $n \times 1$ (ie column matrix) such that

$$(A - \lambda I)X = 0.$$

is called an eigen vector of A . Corresponding to that eigen value λ .

Procedure of finding Eigen vectors:

If λ is an eigen value of A then the corresponding eigen vectors of A will be given by a non-zero vector.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ } n \times 1.$$

Satisfying the equation.

$$\begin{aligned} & (A - \lambda I)X = 0. \quad (\because I \text{ is identity matrix}) \\ \Rightarrow & \boxed{AX = \lambda X} \quad (AX - \lambda IX = 0) \\ & \underline{\underline{AX = \lambda X}}, \end{aligned}$$

Problems ①.

Find the eigen values and eigen vectors of
the matrix $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Solution:

① Eigen values:

i) Characteristic equation: is given by

$$|A - \lambda I| = 0.$$

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0.$$

$$(3-\lambda)[(2-\lambda)(5-\lambda)-0] - 1(0) + 4(0) = 0$$

$$= (3-\lambda)(2-\lambda)(5-\lambda) = 0.$$

$$= (6-3\lambda-2\lambda+\lambda^2)(5-\lambda) = 0.$$

$$= 30-15\lambda-10\lambda+5\lambda^2-6\lambda+9\lambda^2+2\lambda^2-\lambda^3 = 0.$$

$$= -\lambda^3+16\lambda^2-31\lambda+30 = 0.$$

$$= \lambda^3-16\lambda^2+31\lambda-30 = 0.$$

$$\lambda = 2, 3, 5 \text{ (nonrepeated, eigen values)}$$

ii) Eigen vectors:

The eigen vectors of matrix A corresponding to eigen value λ is given by the non-zero solution of equation $(A - \lambda I)x = 0$ where x is eigen vector.

i.e $(A - \lambda I)x = 0$.

$$= \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

when $\lambda = 2$, the corresponding eigen vector is

$$= \begin{bmatrix} 3-2 & 1 & 4 \\ 0 & 2-2 & 6 \\ 0 & 0 & 5-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + x_2 + 4x_3 = 0.$$

$$0 + 0 + 6x_3 = 0.$$

$$\frac{x_1}{6-0} = \frac{-x_2}{6-0} = \frac{x_3}{0} \Rightarrow \frac{x_1}{6} = \frac{x_2}{-6} = \frac{x_3}{0} = k.$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{0} = 6k = k_1$$

$$\frac{x_1}{1} = k_1 ; \frac{x_2}{1} = k_1 ; \frac{x_3}{0} = k_1, \quad (1)$$

First Eigen vector.

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} K_1 \\ -K_1 \\ 0 \end{bmatrix} = K_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ Eigen vector.}$$

when $\lambda = 3$, the corresponding eigen vector.

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 0x_1 + x_2 + 4x_3 = 0$$

$$0x_1 - x_2 + 6x_3 = 0$$

[Cross rule]

or
Rule of cross multiplication

$$\frac{x_1}{6+4} = \frac{-x_2}{0} = \frac{x_3}{0} \Rightarrow \frac{x_1}{10} = \frac{x_2}{0} = \frac{x_3}{0} = k.$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0} = 10k = k_2.$$

$$\frac{x_1}{1} = k_2 ; \frac{x_2}{0} = k_2 \Leftrightarrow \frac{x_3}{0} = k_2.$$

Second Eigen vector.

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} k_2 \\ 0 \\ 0 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ eigenvector.}$$

when $\lambda = 5$, the corresponding eigen vector.

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 + x_2 + 4x_3 = 0$$

$$0x_1 - 3x_2 + 6x_3 = 0$$

$$\frac{x_1}{6+12} = \frac{-x_2}{-12-0} = \frac{x_3}{6-0} \Rightarrow \frac{x_1}{18} = \frac{x_2}{12} = \frac{x_3}{6} = k.$$

$$\frac{x_1}{18} = \frac{x_2}{12} = \frac{x_3}{6} = k.$$

$$\Rightarrow \frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1} = 6k = k_3.$$

$$\frac{x_1}{3} = k_3 ; \frac{x_2}{2} = k_3 ; \frac{x_3}{1} = k_3.$$

$$x_1 = 3k_3 ; x_2 = 2k_3 ; x_3 = k_3.$$

Third eigen vector.

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 3k_3 \\ 2k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Problem ② :

Find the eigen values & eigen vectors of
matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

Solution:

I) Eigen values:

1) Characteristic equation is given by

$$|A - \lambda I| = 0.$$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0.$$

$$= (-2-\lambda) [(1-\lambda)(0-\lambda) - 12] - 2 [2(0-\lambda) - 6] - 3 [-4 + (1-\lambda)]$$

$$= (-2-\lambda) [-\lambda + \lambda^2 - 12] - 2[-2\lambda - 6] - 3[-3 - \lambda]$$

$$= 2\lambda - 2\lambda^2 + 24 + \lambda^2 - \lambda^3 + 12\lambda + 4\lambda + 12 + 9 + 3\lambda$$

$$= -\lambda^3 - \lambda^2 + 21\lambda + 45 = 0.$$

$$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda + 45 = 0.$$

$$(\lambda+3)(\lambda-3)(\lambda-5) = 0$$

$$\lambda = -3, -3, 5 \quad (\text{Repeated eigenvalue})$$

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when $\lambda = -3$, the corresponding eigen vector.

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 - 2R_1, R_3 + R_1$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(Row operation.)

$$= x_1 + 2x_2 - 3x_3 = 0$$

$$x_1 = 0.$$

$$2x_2 - 3x_3 = 0.$$

$$2x_2 = 3x_3 \Rightarrow \frac{x_2}{x_3} = \frac{3}{2}.$$

Hence Eigen vector

$$x_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

$$\text{Now let } x_2 = 0.$$

$$x_1 - 3x_3 = 0$$

$$x_1 = 3x_3$$

$$\frac{x_1}{x_3} = \frac{3}{1},$$

Hence Eigen vector.

$$x_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

II] Eigen vectors:

The eigen vector of matrix A corresponding to eigen value λ is given by the non-zero solution of equation $(A - \lambda I)X = 0$
ie $(A - \lambda I)X = 0$.

$$\Rightarrow \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

when $\lambda = 5$, the corresponding eigen vector is

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -7x_1 + 2x_2 - 3x_3 = 0.$$

$$2x_1 - 4x_2 - 6x_3 = 0.$$

$$\frac{x_1}{-12-12} = \frac{-x_2}{+42+6} = \frac{x_3}{+28-4} \Rightarrow \frac{x_1}{-24} = \frac{x_2}{-48} = \frac{x_3}{24} = k$$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_2}{-2} = \frac{x_3}{+1} = 24k_2 k_1$$

$$\frac{x_1}{-1} = k_1 ; \frac{x_2}{-2} = k_1 ; \frac{x_3}{+1} = k_1$$

$$x_1 = -k_1 ; x_2 = -2k_1 ; x_3 = k_1.$$

First eigen vector.

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 \\ -2k_1 \\ k_1 \end{bmatrix} = k \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Problems: (Problem Set 5.1 → Gilbert Strang)

- 1) Find the eigen values & eigen vectors of the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. Verify that the trace equals the sum of the eigenvalues and the determinant equals their product.

Solution:

If Eigen values:

$$A = \begin{bmatrix} 1 & -1 \\ 2 & u \end{bmatrix}$$

Characteristic Matrix:

$$[A - \lambda I] = \begin{bmatrix} 1 & -1 \\ 2 & u \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Characteristic polynomial:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 \\ 2 & u-\lambda \end{vmatrix} = (1-\lambda)(u-\lambda) + 2 = 4 - \lambda - 4\lambda + \lambda^2 + 2 = \lambda^2 - 5\lambda + 6.$$

Characteristic equation:

$$\therefore |A - \lambda I| = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$\lambda^2 - 3\lambda - 2\lambda + 6 = 0$$

$$\lambda(\lambda - 3) - 2(\lambda - 3) = 0$$

$\lambda = 3, 2$. Eigen values.

trace of matrix A^c $\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

$$\text{Trace } A^c \quad 1 + 4 = 5$$

Eigen values, $\lambda_1 + \lambda_2 = 3 + 2 = 5$

\therefore Trace is equal to the sum of Eigenvalues.

II) Eigen vectors.

$$(A - \lambda I)X = 0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

when $\lambda = 3$.

$$\Rightarrow \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} -2x_1 - x_2 &= 0 \Rightarrow x_2 = -2x_1 \\ 2x_1 + x_2 &= 0 \end{aligned}$$

$$\frac{x_1}{x_2} = \frac{1}{-2}$$

$$x_1 = c x_2$$

Hence eigenvector
 $x_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

11(4) when $\lambda = 2$.

$$\begin{bmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \xrightarrow{\text{Row reduction}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 - x_2 = 0 \Rightarrow x_2 = -x_1$$

$$2x_1 + 2x_2 = 0.$$

Hence Eigen vector

$$x_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$|\text{Det } A| = \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix}$$

$$= 4 + 2 = 6.$$

Product of eigen values $\lambda_1 \lambda_2 = 3 \times 2 = 6$.

Determinant is equal to the product of eigen values.

Problem set 5.1 \Rightarrow Gurbat Shringi

- (3) If we shift to $A - ET$, what are the eigenvalues & eigenvectors and how are they related to those of A ?

$$B = A - ET \quad \begin{bmatrix} 6 & -1 \\ 2 & -3 \end{bmatrix}$$

Solution:

$$B = A - ET$$

Find eigenvalues & eigenvectors of A , so we
to find matrix A .

$$\Rightarrow A = B + ET \Rightarrow$$

$$= \begin{bmatrix} 6 & -1 \\ 2 & -3 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -1 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

Now compute the eigen value of A.

$$(A - \lambda I) = 0.$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (1-\lambda)(4-\lambda) + 2 = 0$$

$$\Rightarrow 4 - \lambda - 4\lambda + \lambda^2 + 2 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda - 6 = 0.$$

$$\lambda_1 = 3, \quad \lambda_2 = 2.$$

Now we will determine eigenvalues of B.

$$(B - \lambda I) = 0$$

$$\begin{pmatrix} -6-\lambda' & -1 \\ 2 & -3-\lambda' \end{pmatrix} = 0.$$

$$= (-6-\lambda')(-3-\lambda') + 2 = 0.$$

$$= +18 + 6\lambda' + 3\lambda'^2 + 8\lambda'^2 + 2 = 0.$$

$$= \lambda'^2 + 9\lambda' + 20 = 0.$$

$$= \lambda'_1 = -4, \quad \lambda'_2 = -5.$$

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Now let us compute the Eigen vector of A.

$$(A - \lambda I) \mathbf{x} = 0$$

$$\begin{bmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

when $\lambda_1 = 3$.

$$\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-2x_1 - x_2 = 0$$

$$2x_1 + x_2 = 0.$$

$$-2x_1 = x_2$$

$$\frac{x_1}{x_2} = \frac{-1}{-2} \Rightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix} \rightarrow \text{first Eigen vector.}$$

when $\lambda_2 = 2$.

$$\begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$$-x_1 - x_2 = 0$$

$$2x_1 + 2x_2 = 0.$$

$$+x_1 = -x_2$$

$$\frac{x_1}{x_2} = \frac{-1}{+1} \Rightarrow \begin{bmatrix} -1 \\ +1 \end{bmatrix} \rightarrow \text{Second Eigen vector.}$$

III^w compute the Eigen vector of B.

$$(B - \lambda I) \mathbf{x} = 0$$

$$= \begin{bmatrix} -6-\lambda^1 & -1 \\ 2 & -3-\lambda^1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (-6-\lambda^1)(-3-\lambda^1)$$

when $\lambda_1 = -4$

$$= \begin{bmatrix} -6+4 & -1 \\ 2 & -3+4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1 \\ 2 & +1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= -2x_1 - x_2 = 0 \Rightarrow -2x_1 = x_2$$

$$2x_1 + x_2 = 0 \quad \frac{x_1}{x_2} = \frac{1}{-2} \Rightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix} \Rightarrow \text{first Eigen vector.}$$

III^y when $\lambda_2 = -5$

$$\Rightarrow \begin{bmatrix} -6+5 & -1 \\ 2 & -3+5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 - x_2 = 0 \Rightarrow x_1 = -x_2$$

$$2x_1 + 2x_2 = 0 \quad \frac{x_1}{x_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \text{second Eigen vector.}$$

Since the eigen vectors of A & eigen vectors of B are same, they are related.

Diagonalization of a Matrix:-

- * It is a process of reduction of matrix A to a diagonal form D.
- * If A is related to D by a similarity transformation such that

$$P^{-1}AP = D.$$

then A is reduced to the diagonal matrix D through modal matrix P.

Important Notes:-

- 1) The matrix P which diagonalises A is called modal matrix of A.
- 2) Modal matrix is formed by grouping the eigen vectors of A into square matrix.
- 3) The resulting diagonal matrix D is called spectral matrix of A.
- 4) Spectral matrix has the eigen values of A as its diagonal elements.
- 5) The transformation of matrix A to $P^{-1}AP$ is called "similarity transformation".

Problem:

1) Diagonalize the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Solution:

1) characteristic equation of matrix A is

$$|A - \lambda I| = \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (6-\lambda) \left[(3-\lambda)^2 - 1 \right] + 2 \left[(-6+2\lambda)+2 \right] + 2 \left[2 - (6-2\lambda) \right] = 0.$$

$$\Rightarrow (6-\lambda) [9+\lambda^2-6\lambda-1] + 2[2\lambda-4] + 2[+2\lambda-4] = 0.$$

$$\Rightarrow (6-\lambda) [\lambda^2-6\lambda+8] + 4\lambda - 8 + 4\lambda - 8 = 0$$

$$\Rightarrow 6\lambda^2 - 36\lambda + 48 - \lambda^3 + 6\lambda^2 - 8\lambda + 4\lambda - 8 + 4\lambda - 8 = 0,$$

$$\Rightarrow -\lambda^3 + 12\lambda^2 - 36\lambda - 32 = 0.$$

$$\Rightarrow \lambda^3 - 12\lambda^2 + 36\lambda + 32 = 0.$$

$\lambda = 2, 2, 8$ - Repeated Eigen values.

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2) Eigen vectors;

For $\lambda = 8$, the eigen vector:

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & - & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 - 2x_2 + 2x_3 = 0.$$

$$-2x_1 - 5x_2 - x_3 = 0.$$

$$\frac{x_1}{+2+10} = \frac{-x_2}{2+4} = \frac{x_3}{10-4} \Rightarrow \frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}.$$

$$\frac{x_1}{12} = k_1 ; \frac{x_2}{-6} = k_1 ; \frac{x_3}{6} = k_1$$

$$\frac{x_1}{6} = 2k_1 ; \frac{x_2}{-1} = -k_1 ; \frac{x_3}{1} = k_1$$

first eigen vector.

$$x_1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k_1 \\ -k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \text{ Eigen vectors.}$$

For $\lambda = 2$, the eigen vector.

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 4x_1 - 2x_2 + 2x_3 = 0.$$

$$-2x_1 + x_2 - x_3 = 0. \Rightarrow$$

~~R2 + R1~~

$$R_2 + \frac{1}{2}R_1$$

$$R_3 - \frac{1}{2}R_1$$

$$\begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\Rightarrow 4x_1 - 2x_2 + 2x_3 = 0.$$

Let $x_1 \neq 0$.

$$2x_2 = 2x_3 \Rightarrow \frac{x_2}{x_3} = 1 \Rightarrow \text{Eigen Vector } x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Let $x_2 \neq 0$.

$$4x_1 = -2x_3 \Rightarrow \frac{x_1}{x_3} = -\frac{1}{2} \Rightarrow \text{Eigen Vector } x_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

3) modal matrix P:-

$$\therefore P = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

↑ ↑ ↑
 Eigen Eigen Eigen
 vector vector vector
 $\lambda=2$ $\lambda=2$ $\lambda=2$

$$P^{-1} = \frac{\text{adj}(P)}{|P|}$$

$$|P| = \begin{vmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 2(2) - 0 - 1(-1 - 1)$$

$$= 4 + 2 = 6$$

$$|P| = \underline{\underline{6}}$$

$$\text{adj}(P) = \begin{bmatrix} +2 & -1 & +1 \\ -(-2) & +5 & -(-1) \\ +(-2) & -2 & +2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & +1 \\ +2 & 5 & +1 \\ -2 & -2 & +2 \end{bmatrix}$$

$$\therefore P^{-1} = \frac{\text{adj} P}{|P|}$$

$$P^{-1} = \frac{1}{6} \begin{bmatrix} 2 & -1 & +1 \\ 2 & 5 & +1 \\ -2 & -2 & +2 \end{bmatrix}$$

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Now $P^{-1}AP = D$.

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Now

$$P^{-1}AP = \frac{1}{6} \begin{bmatrix} 2 & -1 & +1 \\ 2 & 5 & 1 \\ -2 & -2 & +2 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{6} \begin{bmatrix} 16 & -8 & 8 \\ 4 & 10 & 2 \\ -4 & -4 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 48 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D \Rightarrow \text{Diagonal matrix or Spectral matrix.}$$

Hence $P = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Problem 2:

Determine the eigen values and eigen vectors

of matrix $A = \begin{bmatrix} 5 & 1 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{bmatrix}$. Hence find matrix.

P such that $P^{-1}AP$ is a diagonal matrix.

Solution:

(1) Characteristic equation of matrix A is

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 1 & -5 \\ 0 & 4-\lambda & -1 \\ 2 & 8 & -3-\lambda \end{vmatrix} = 20$$

$$\Rightarrow (5-\lambda)[(4-\lambda)(-3-\lambda)+8] - 7[0+2] - 5[0-2(4-\lambda)] = 20$$

$$\Rightarrow (5-\lambda)[-13-4\lambda+3\lambda+\lambda^2+8] = 14 - 5[-8+2\lambda] = 20,$$

$$\Rightarrow (5-\lambda)[\lambda^2-\lambda-4] = 14 + 40 - 10\lambda = 0,$$

$$\Rightarrow 5\lambda^2 - 5\lambda - 20 - \lambda^3 + \lambda^2 + 4\lambda - 14 + 40 - 10\lambda = 0,$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 + 11\lambda - 6 = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 - 11\lambda + 6 = 0$$

$$\Rightarrow \lambda = 1, 2, 3 \Rightarrow \text{Eigen values.}$$

② Eigen vector:

when $\lambda = 1$, the corresponding eigen vector.

$$\Rightarrow \begin{bmatrix} 4 & 7 & -5 \\ 0 & 3 & -1 \\ 2 & 8 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 4x_1 + 7x_2 - 5x_3 = 0 \\ 0 + 3x_2 - x_3 = 0.$$

$$\frac{x_1}{-7+15} = \frac{-x_2}{-4} = \frac{x_3}{12} = \frac{x_1}{+8} = \frac{x_2}{+4} = \frac{x_3}{12} = k.$$

$$\frac{x_1}{+2} = \frac{x_2}{+1} = \frac{x_3}{3} = 4k. = k_1$$

$$\frac{x_1}{2} = k_1 ; \frac{x_2}{1} = k_1 ; \frac{x_3}{3} = k_1$$

$$x_1 = 2k_1 ; x_2 = k_1 ; x_3 = 3k_1$$

$$x_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k_1 \\ k_1 \\ 3k_1 \end{bmatrix} = k_1 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \rightarrow \text{first eigen vector}$$

when $\lambda = 2$, the corresponding eigen vector

$$\begin{bmatrix} 3 & 7 & -5 \\ 0 & 2 & -1 \\ 2 & 8 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x_1 + 7x_2 - 5x_3 = 0 \\ 0 + 2x_2 - x_3 = 0$$

$$\frac{x_1}{-7+10} = \frac{-x_2}{-3-0} = \frac{x}{6-0} \Rightarrow \frac{x_1}{3} = \frac{x_2}{3} = \frac{x}{6} = k.$$

$$\frac{x_1}{1} ; \frac{x_2}{1} ; \frac{x_3}{2} = 2k = k_2.$$

$$x_1 = k_2; x_2 = k_2; x_3 = 2k_2$$

$$x_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_2 \\ k_2 \\ 2k_2 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \rightarrow 2^{\text{nd}} \text{ eigen vector},$$

when $\lambda = 3$; the corresponding eigen vector.

$$\begin{bmatrix} 2 & 1 & -5 \\ 0 & 1 & -1 \\ 2 & 8 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} 2x_1 + 7x_2 - 5x_3 &= 0 \\ 0 + x_2 - x_3 &= 0 \end{aligned}$$

$$\frac{x_1}{-7+5} = \frac{-x_2}{-2-0} = \frac{x_3}{2-0} \Rightarrow \frac{x_1}{-2} = \frac{-x_2}{-2} = \frac{x_3}{2} = k.$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1} = 2k = k_3.$$

$$x_1 = k_3; x_2 = k_3; x_3 = k_3.$$

$$x_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_3 \\ k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \rightarrow 3^{\text{rd}} \text{ eigen vector}.$$

③ Modal matrix P:

$$P = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \quad P^{-1} = \frac{\text{adj}(P)}{|P|}$$

$\downarrow \downarrow \downarrow$
 $\lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = 3$

$$|P| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix}$$

$$|P| = 2(1-2) - 1(1-3) - 1(2-3),$$

$$= 2(-1) - 1(-2) - 1(-1),$$

$$= -2 + 2 + 1$$

$$|P| = 1$$

Now find adj(P).

$$\text{adj}(P) = \begin{bmatrix} +(+1) & -3 & +2 \\ -(-2) & +5 & -3 \\ +(-1) & -1 & +1 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 2 \\ +2 & 5 & -3 \\ -1 & -1 & +1 \end{bmatrix}$$

$$\therefore P^{-1} = \frac{\text{adj}(P)}{|P|} = \begin{bmatrix} -1 & -3 & 2 \\ 2 & 5 & -3 \\ -1 & -1 & +1 \end{bmatrix}$$

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Now $P^{-1}AP = D$.

$$D_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

\Rightarrow Now

$$P^T A P = \begin{bmatrix} -1 & -3 & 2 \\ 2 & 5 & -3 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow D \text{ diagonal matrix } \& \text{ spectral matrix.}$$

Hence $P_2 = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

Positive definite matrices.

Symmetric matrices:

$$(1) A = A^T$$

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}, \quad A^T = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$$

Definition:

A symmetric matrix $n \times n$ real matrix M is said to be positive definite matrix such that $z^T M z > 0$ for every non zero column vector z .

(2) 'n' real numbers,

z^T transpose of z .

i.e M positive definite $\Rightarrow x^T M x > 0$ for all $x \in R^n$.

Eg:

$$z = \begin{bmatrix} a \\ b \end{bmatrix}, \quad M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$z^T [a, b]$$

$$\Rightarrow z^T M z = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a+b & a+2b \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\Rightarrow 2a^2 + b^2 + a^2 + 2b^2 \Rightarrow 3a^2 + 3b^2 > 0 \quad \forall (a, b) \in R^2 - (0, 0)$$

(for all non zero's of a, b)

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$$\text{Ex:- } M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad Z = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

Definition:

$$Z^T M Z > 0.$$

$$Z = \begin{bmatrix} 3 \\ -4 \end{bmatrix}, Z^T = \begin{bmatrix} 3 & -4 \end{bmatrix}$$

ie

$$\begin{bmatrix} 3 & -4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 6-4 & 3-8 \\ 3-8 & 16-16 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -5 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$= 6 + 20 = \underline{\underline{26}} > 0.$$

Hence M is a positive definite matrix.

Properties of positive definite matrix:-

1) Identity matrix is positive definite matrix

$$\text{Eg: } I_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$z = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, z^T = \begin{bmatrix} a, b, c \end{bmatrix}.$$

$$z^T I z > 0.$$

$$= \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= a^2 + b^2 + c^2 > 0$$

• Sum of squares > 0 if $(a, b, c) \in \mathbb{R}^3 - (0, 0, 0)$.

(strictly positive, not even zero).

Singular Value Decomposition: (SVD)

* SVD is a method of decomposing a rectangular matrix into three matrices

$$A = U \Sigma V^T = (\text{Orthogonal}) (\text{Diagonal}) (\text{Orthogonal}),$$

where A = Input matrix which is of size $m \times n$.

U & V are orthogonal matrices

& Σ is a diagonal matrix.

Orthogonal matrix.

$$AA^T = A^T A = I$$

SVD Properties:

- ① The decomposed matrices are unique.
- ② U & V are column orthonormal.
- ③ The columns are U & V orthogonal.
- ④ The values in ' Σ ' matrix are called singular values and they are positive and sorted in decreasing order ($\sigma_1 \geq \sigma_2 \geq \dots \geq 0$)

Steps to find SVD:

- 1) $A^T A$.
- 2) Find eigen values of $A^T A$ and arrange them in descending order.
- 3) Find the ~~secon~~ singular values.
The singular values $\sigma_1 = \sqrt{\lambda_1}$; $\sigma_2 = \sqrt{\lambda_2}$
- 4) Find the eigen vectors for each eigen value.
- 5) calculate U, V & Σ matrices.

where 1) $U = \left\{ \frac{1}{\sigma_1} A v_1, \frac{1}{\sigma_2} A v_2, \dots \right\}$

$$= \left\{ \frac{1}{\sigma_1} A v_1, \frac{1}{\sigma_2} A v_2 \right\}.$$

2) $V = [V_1, V_2] \Rightarrow$

$$\begin{bmatrix} \frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|} \end{bmatrix}$$

where x_1, x_2 = eigen vector.

$$\|x_1\| = \sqrt{x_1^2 + x_2^2}, \|x_2\| = \sqrt{x_1^2 + x_2^2}$$

- 3) Obtain diagonal matrix Σ

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

Note:- Size of diagonal matrix Σ = size of given matrix.

Problem:

- 1) Find the singular value decomposition of the matrix $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

Solution.

Step 1: Find $A^T A^*$

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow A^T A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 \\ 6 & 9+4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$$

Step 2: Find the eigen values of $A^T A$ and arrange them in descending order.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & 6 \\ 6 & 13-\lambda \end{vmatrix} = 0. \quad (4-\lambda)(13-\lambda) - 36 = 0 \\ 52 - 4\lambda - 13\lambda + \lambda^2 - 36 = 0. \\ \lambda^2 - 17\lambda + 16 = 0.$$

$\lambda_1 = 16, \lambda_2 = 1.$ (Arrange in descending order).

$$\lambda_1 = 16, \lambda_2 = 1.$$

Step 3: Find the singular values.

The singular value $\sigma_1 = \sqrt{\lambda_1} \Rightarrow \sigma_1 = \sqrt{16}$, $\sigma_2 = \sqrt{\lambda_2} \Rightarrow \sigma_2 = \sqrt{1}$.

$$\sigma_1 = \sqrt{16}, \sigma_2 = \sqrt{1}$$

$$\sigma_1 = 4, \sigma_2 = 1.$$

Step 4: Find the eigen vector for each eigen values.

$$\text{ie } (A - \lambda I) X = 0.$$

$$\begin{bmatrix} 4-\lambda & 6 \\ 6 & 13-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

First eigen vector:-

$$\text{when } \lambda = 16.$$

$$= \begin{bmatrix} 4-16 & 6 \\ 6 & 13-16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -12x_1 + 6x_2 = 0$$

$$6x_1 - 3x_2 = 0.$$

$$\Rightarrow \sqrt{12}x_1 = \sqrt{6}x_2$$

$$\Rightarrow 2x_1 = x_2$$

$$x_1 = \frac{x_1}{x_2} = \frac{1}{2} = v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \text{first Eigen vector.}$$

Second eigen vector:

when $\lambda = 1$.

$$\begin{bmatrix} 4-1 & 6 \\ 6 & 13-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= 3x_1 + 6x_2 = 0.$$

$$6x_1 + 12x_2 = 0.$$

$$3x_1 + 6x_2$$

$$x_1 + 2x_2$$

$$x_2 = \frac{x_1}{x_2} = -\frac{2}{1} \Rightarrow v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \rightarrow \text{Second eigen vector.}$$

Step 5:- Calculate U, V, & Σ matrices.

where

$$\textcircled{1} \quad U = \left\{ \frac{1}{\sigma_1} Av_1, \quad ; \quad \frac{1}{\sigma_2} Av_2 \right\}$$

where $\sigma_1 = \sqrt{5}$

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$$

$$v_1 = \frac{x_1}{\|x_1\|} \Rightarrow \|x_1\| = \sqrt{x_1^2 + x_2^2}$$

$$= \sqrt{1^2 + 2^2}$$

$$\|x_1\| = \sqrt{1^2 + 4} = \sqrt{5}$$

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$$\therefore v_1 = \frac{x_1}{\|x_1\|} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

||| 4

$$v_2 = \frac{|x_2|}{\|x_2\|} \Rightarrow \|x_2\| = \sqrt{x_1^2 + x_2^2} \\ = \sqrt{-2^2 + 1^2} \\ = \sqrt{4^2 + 1} \\ = \sqrt{5}$$

$$\therefore v_2 = \frac{|x_2|}{\|x_2\|} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

Now $U = \left[\frac{1}{\sigma_1} \Lambda v_1, \frac{1}{\sigma_2} \Lambda v_2 \right]$

$$U_1 = \frac{1}{4} \left[\begin{array}{cc} 2 & 3 \\ 0 & 2 \end{array} \right] \begin{bmatrix} 1/\sqrt{5}, \\ 2/\sqrt{5} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \frac{2}{\sqrt{5}} + \frac{6}{\sqrt{5}} \\ \frac{0}{\sqrt{5}} + \frac{4}{\sqrt{5}} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \frac{8}{\sqrt{5}} \\ \frac{4}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

||| 4 when $\sigma_2 = 1$

$$v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$U_2 = \left[\frac{1}{\sqrt{2}} A V_2 \right]$$

$$\Rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$\Rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{4}{\sqrt{5}} + \frac{3}{\sqrt{5}} \\ 0 + \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \quad \text{--- (1)}$$

$$\begin{aligned} \textcircled{2} \quad V_2 &= \begin{bmatrix} \frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|} \end{bmatrix} \\ &= \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \quad \text{--- (2)} \end{aligned}$$

\textcircled{3} Σ = Diagonal matrix Σ .

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{--- (3)}$$

$$\therefore A_2 = U \Sigma V^T$$

$$\Rightarrow A = U \Sigma V^T$$

$$= \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$\Rightarrow A = \underline{U \Sigma V^T}$$

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