CHAPTER 2

TIME RESPONSE ANALYSIS

2.1 TIME RESPONSE

The time response of the system is the output of the closed loop system as a function of time. It is denoted by c(t). The time response can be obtained by solving the differential equation governing the system. Alternatively, the response c(t) can be obtained from the transfer function of the system and the input to the system.

The closed loop transfer function, $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)\,H(s)} = M(s)$ .....(2.1)

The Output or Response in s-domain, C(s) is given by the product of the transfer function and the input, R(s). On taking inverse Laplace transform of this product the time domain response, c(t) can be obtained.

Response in s-domain, $C(s) = R(s)\,M(s)$ .....(2.2)

Response in time domain, $c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\{R(s) \times M(s)\}$ .....(2.3)

where, $M(s) = \dfrac{G(s)}{1+G(s)H(s)}$

The time response of a control system consists of two parts : *the transient and the steady state response*. The transient response is the response of the system when the input changes from one state to another. The steady state response is the response as time, **t** approaches infinity.
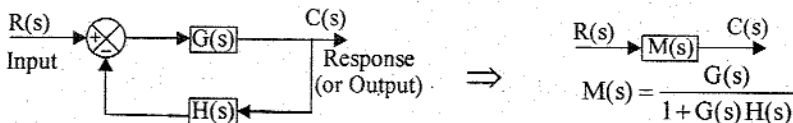

Fig 2.1 : Closed loop system.

2.2 TEST SIGNALS

The knowledge of input signal is required to predict the response of a system. In most of the systems the input signals are not known ahead of time and also it is difficult to express the input signals mathematically by simple equations. The characteristics of actual input signals are a sudden shock, a sudden change, a constant velocity and a constant acceleration. Hence test signals which resembles these characteristics are used as input signals to predict the performance of the system. The commonly used test input signals are impulse, step, ramp, acceleration and sinusoidal signals.

The standard test signals are,

1. a) Step signal   2. a) Ramp signal   3. a) Parabolic signal
   b) Unit step signal   b) Unit ramp signal   b) Unit parabolic signal
4. Impulse signal   5. Sinusoidal signal.

Since the test signals are simple functions for time, they can be easily generated in laboratories. The mathematical and experimental analysis of control systems using these signals can be carried out easily. The use of the test signals can be justified because of a correlation existing between the response characteristics of a system to a test input signal and capability of the system to cope with actual input signals.

STEP SIGNAL

The step signal is a signal whose value changes from zero to A at t = 0 and remains constant at A for t > 0. The step signal resembles an actual steady input to a system. A special case of step signal is unit step in which **A** is unity.

The mathematical representation of the step signal is,

$$r(t) = 1 \; ; \; t \geq 0$$
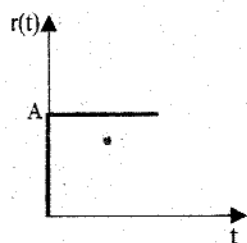$$= 0 \; ; \; t < 0 \qquad \qquad .....(2.4)$$

Fig 2.2 : Step signal.

RAMP SIGNAL

The ramp signal is a signal whose value increases linearly with time from an initial value of zero at t = 0. The ramp signal resembles a constant velocity input to the system. A special case of ramp signal is unit ramp signal in which the value of **A** is unity.

The mathematical representation of the ramp signal is,

$$r(t) = A t \; ; \; t \geq 0$$
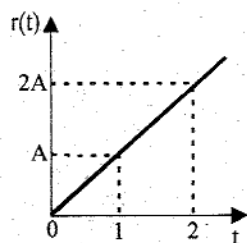$$= 0 \quad ; \; t < 0 \qquad \qquad .....(2.5)$$

Fig 2.3 : Ramp signal.

PARABOLIC SIGNAL

In parabolic signal, the instantaneous value varies as square of the time from an initial value of zero at t = 0. The sketch of the signal with respect to time resembles a parabola. The parabolic signal resembles a constant acceleration input to the system. A special case of parabolic signal is unit parabolic signal in which A is unity.

The mathematical representation of the parabolic signal is,

$$r(t) = \frac{At^2}{2} \; ; \; t \geq 0$$
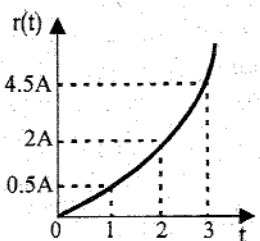$$= 0 \quad ; \; t < 0 \qquad \qquad .....(2.6)$$

Fig 2.4 : Parabolic signal.

Note : *Integral of step signal is ramp signal. Integral of ramp signal is parabolic signal.*

IMPULSE SIGNAL

A signal of very large magnitude which is available for very short duration is called *impulse signal.* Ideal impulse signal is a signal with infinite magnitude and zero duration but with an area of A. The unit impulse signal is a special case, in which A is unity.

The impulse signal is denoted by δ(t) and mathematically it is expressed as,

$$\delta(t) = \infty; \; t = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(t) dt = A$$
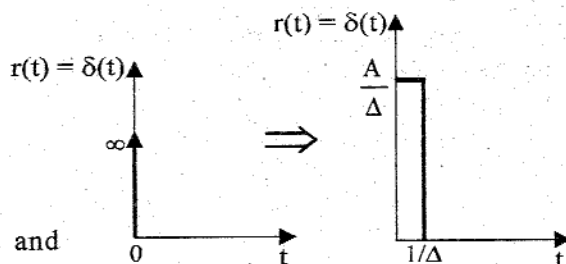$$= 0; \; t \neq 0 \qquad \qquad .....(2.7)$$

Fig 2.5 : Impulse signal.

Since a perfect impulse cannot be achieved in practice it is usually approximated by a pulse of small width but with area, A. Mathematically an impulse signal is the derivative of a step signal. Laplace transform of the impulse function is unity.

TABLE 2-1 : Standard Test Signals

Name of the signal	Time domain equation of signal, r(t)	Laplace transform of the signal, R(s)
Step	A	$\dfrac{A}{s}$
Unit step	1	$\dfrac{1}{s}$
Ramp	At	$\dfrac{A}{s^2}$
Unit ramp	t	$\dfrac{1}{s^2}$
Parabolic	$\dfrac{At^2}{2}$	$\dfrac{A}{s^3}$
Unit parabolic	$\dfrac{t^2}{2}$	$\dfrac{1}{s^3}$
Impulse	$\delta(t)$	1

2.3 IMPULSE RESPONSE

The response of the system, with input as impulse signal is called *weighing function* or *impulse response* of the system. It is also given by the inverse Laplace transform of the system transfer function, and denoted by m(t).

$$\text{Impulse response, } m(t) = \mathcal{L}^{-1}\{R(s)\,M(s)\} = \mathcal{L}^{-1}\{M(s)\} \qquad .....(2.8)$$

$$\text{where, } M(s) = \frac{G(s)}{1+G(s)H(s)} \qquad \boxed{R(s) = 1, \text{ for impulse}}$$

Since impulse response (or weighing function) is obtained from the transfer function of the system, it shows the characteristics of the system. Also the response for any input can be obtained by convolution of input with impulse response.

2.4 ORDER OF A SYSTEM

The input and output relationship of a control system can be expressed by n^{th} order differential equation shown in equation (2.9).

$$a_0 \frac{d^n}{dt^n}p(t) + a_1 \frac{d^{n-1}}{dt^{n-1}}p(t) + a_2 \frac{d^{n-2}}{dt^{n-2}}p(t) + ... + a_{n-1}\frac{d}{dt}p(t) + a_n\, p(t) = b_0 \frac{d^m}{dt^m}q(t)$$

$$+ b_1 \frac{d^{m-1}}{dt^{m-1}}q(t) + b_2 \frac{d^{m-2}}{dt^{m-2}}q(t) + ... + b_{m-1}\frac{d}{dt}q(t) + b_m\, q(t) \qquad .....(2.9)$$

where, $p(t)$ = Output / Response ; $q(t)$ = Input / Excitation.

The order of the system is given by the order of the differential equation governing the system. If the system is governed by n^{th} order differential equation, then the system is called *n^{th} order system.*

Alternatively, the order can be determined from the transfer function of the system. The transfer function of the system can be obtained by taking Laplace transform of the differential equation governing the system and rearranging them as a ratio of two polynomials in **s**, as shown in equation (2.10).

$$\text{Transfer function, } T(s) = \frac{P(s)}{Q(s)} = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \dots\dots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots\dots + a_{n-1} s + a_n} \qquad \dots(2.10)$$

where, $P(s) =$ Numerator polynomial

$\qquad Q(s) =$ Denominator polynomial

The order of the system is given by the maximum power of **s** in the denominator polynomial, $Q(s)$.

Here, $Q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots\dots\dots\dots + a_{n-1} s + a_n$.

Now, **n** is the order of the system

When n = 0, the system is zero order system.

When n = 1, the system is first order system.

When n = 2, the system is second order system and so on.

> **Note :** The order can be specified for both open loop system and closed loop system.

The numerator and denominator polynomial of equation (2.10) can be expressed in the factorized form as shown in equation (2.11).

$$T(s) = \frac{P(s)}{Q(s)} = \frac{(s+z_1)(s+z_2)\dots\dots(s+z_m)}{(s+p_1)(s+p_2)\dots\dots(s+p_n)} \qquad \dots(2.11)$$

where, $\quad z_1, z_2, \dots\dots z_m$ are zeros of the system.

$\qquad\qquad p_1, p_2, \dots\dots p_n$ are poles of the system.

Now, the value of **n** gives the number of poles in the transfer function. Hence the order is also given by the number of poles of the transfer function.

> **Note :** The zeros and poles are critical value, of s, at which the function T(s) attains extreme values 0 or ∞. When s takes the value of a zero, the function T(s) will be zero. When s takes the value of a pole, the function T(s) will be infinite.

2.5 REVIEW OF PARTIAL FRACTION EXPANSION

The time response of the system is obtained by taking the inverse Laplace transform of the product of input signal and transfer function of the system. Taking inverse Laplace transform requires the knowledge of partial fraction expansion. In control systems three different types of transfer function are encountered. They are,

Case 1 : Functions with separate poles.

Case 2 : Functions with multiple poles.

Case 3 : Functions with complex conjugate poles.

The partial fraction of all the three cases are explained with an example.

Case 1 : When the transfer function has distinct poles

$$\text{Let, } T(s) = \frac{K}{s(s+p_1)(s+p_2)}$$

By partial fraction expansion, T(s) can be expressed as,

$$T(s) = \frac{K}{s(s+p_1)(s+p_2)} = \frac{A}{s} + \frac{B}{s+p_1} + \frac{C}{s+p_2}$$

The residues A, B and C are given by,

$$A = T(s) \times s\Big|_{s=0} \qquad B = T(s) \times (s+p_1)\Big|_{s=-p_1} \qquad C = T(s) \times (s+p_2)\Big|_{s=-p_2}$$

Example

Let, $T(s) = \dfrac{2}{s(s+1)(s+2)}$

By partial fraction expansion, T(s) can be expressed as,

$$T(s) = \frac{2}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

A is obtained by multiplying T(s) by s and letting s = 0.

$$A = T(s) \times s\Big|_{s=0} = \frac{2}{s(s+1)(s+2)} \times s\Big|_{s=0} = \frac{2}{(s+1)(s+2)}\Big|_{s=0} = \frac{2}{1 \times 2} = 1$$

B is obtained by multiplying T(s) by (s +1) and letting s = −1.

$$B = T(s) \times (s+1)\Big|_{s=-1} = \frac{2}{s(s+1)(s+2)} \times (s+1)\Big|_{s=-1} = \frac{2}{s(s+2)}\Big|_{s=-1} = \frac{2}{-1(-1+2)} = -2$$

C is obtained by multiplying T(s) by (s +2) and letting s = −2.

$$C = T(s) \times (s+2)\Big|_{s=-2} = \frac{2}{s(s+1)(s+2)} \times (s+2)\Big|_{s=-2} = \frac{2}{s(s+1)}\Big|_{s=-2} = \frac{2}{-2(-2+1)} = +1$$

$$\therefore T(s) = \frac{2}{s(s+1)(s+2)} = \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}$$

Case 2 : When the transfer function has multiple poles

Let, $T(s) = \dfrac{K}{s(s+p_1)(s+p_2)^2}$

By partial fraction expansion, T(s) can be expressed as,

$$T(s) = \frac{K}{s(s+p_1)(s+p_2)^2} = \frac{A}{s} + \frac{B}{s+p_1} + \frac{C}{(s+p_2)^2} + \frac{D}{(s+p_2)}$$

The residues A, B, C and D are given by,

$$A = T(s) \times s\Big|_{s=0} \qquad\qquad B = T(s) \times (s+p_1)\Big|_{s=-p_1}$$

$$C = T(s) \times (s+p_2)^2\Big|_{s=-p_2} \qquad D = \frac{d}{ds}\Big[T(s) \times (s+p_2)^2\Big]\Big|_{s=-p_2}$$

Example

Let, $T(s) = \dfrac{2}{s(s+1)(s+2)^2}$

By partial fraction expansion, T(s) can be expressed as,

$$T(s) = \frac{K}{s(s+1)(s+2)^2} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)}$$

A is obtained by multiplying T(s) by s and letting s = 0.

$$A = T(s) \times s \Big|_{s=0} = \frac{2}{s(s+1)(s+2)^2} \times s \Big|_{s=0} = \frac{2}{(s+1)(s+2)^2} \Big|_{s=0} = \frac{2}{1 \times 2^2} = 0.5$$

B is obtained by multiplying T(s) by (s +1) and letting s = -1.

$$B = T(s) \times (s+1) \Big|_{s=-1} = \frac{2}{s(s+1)(s+2)^2} \times (s+1) \Big|_{s=-1} = \frac{2}{s(s+2)^2} \Big|_{s=-1} = \frac{2}{-1(-1+2)^2} = -2$$

C is obtained by multiplying T(s) by $(s+2)^2$ and letting s = -2.

$$C = T(s) \times (s+2)^2 \Big|_{s=-2} = \frac{2}{s(s+1)(s+2)^2} \times (s+2)^2 \Big|_{s=-2} = \frac{2}{s(s+1)} \Big|_{s=-2} = \frac{2}{-2(-2+1)} = 1$$

D is obtained by differentiating the product T(s) $(s+2)^2$ with respect to s and then letting s = -2.

$$D = \frac{d}{ds} \Big[T(s) \times (s+2)^2 \Big] \Big|_{s=-2} = \frac{d}{ds} \Big[\frac{2}{s(s+1)} \Big] \Big|_{s=-2} = \frac{-2(2s+1)}{s^2(s+1)^2} \Big|_{s=-2} = \frac{-2(2(-2)+1)}{(-2)^2(-2+1)^2} = +1.5$$

$$\therefore T(s) = \frac{2}{s(s+1)(s+2)^2} = \frac{0.5}{s} - \frac{2}{s+1} + \frac{1}{(s+2)^2} + \frac{1.5}{s+2}$$

Case 3 : When the transfer function has complex conjugate poles

Let, $T(s) = \dfrac{K}{(s+p_1)(s^2+bs+c)}$

By partial fraction expansion, T(s) can be expressed as,

$$T(s) = \frac{K}{(s+p_1)(s^2+bs+c)} = \frac{A}{s+p_1} + \frac{Bs+C}{s^2+bs+c} \qquad .....(2.12)$$

The residue A is given by, $A = T(s) \times (s+p_1)\Big|_{s=-p_1}$

The residues B and C are solved by cross multiplying the equation (2.12) and then equating the coefficient of like power of **s**.

Finally express T(s) as shown below,

$$T(s) = \frac{A}{s+p_1} + \frac{Bs+C}{s^2+bs+c} \qquad \boxed{(x+y)^2 = x^2 + 2xy + y^2}$$

Let us express, $s^2 + bs$, in the form of $(x+y)^2$. This will require addition and subtraction of an extra term $(b/2)^2$.

$$\therefore T(s) = \frac{A}{s+p_1} + \frac{Bs+C}{s^2 + 2 \times \frac{b}{2}s + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2} = \frac{A}{s+p_1} + \frac{Bs+C}{\left(s+\frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)}$$

$$= \frac{A}{s+p_1} + \frac{Bs}{\left(s+\frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)} + \frac{C}{\left(s+\frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)}$$

Example

Let, $T(s) = \dfrac{1}{(s+2)(s^2+s+1)}$

By partial fraction expansion,

$$T(s) = \frac{1}{(s+2)(s^2+s+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+s+1}$$

A is obtained by multiplying T(s) by (s +2) and letting s = −2.

$$\therefore A = T(s) \times (s+2)\Big|_{s=-2} = \frac{1}{(s+2)(s^2+s+1)} \times (s+2)\Big|_{s=-2} = \frac{1}{(-2)^2-2+1} = \frac{1}{3}$$

To solve B and C, cross multiply the following equation and substitute the value of A. Then equate the like power of s.

$$\frac{1}{(s+2)(s^2+s+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+s+1}$$

$$1 = A(s^2+s+1) + (Bs+C)(s+2)$$

$$1 = \frac{1}{3}(s^2+s+1) + Bs^2 + 2Bs + Cs + 2C$$

$$1 = \frac{s^2}{3} + \frac{s}{3} + \frac{1}{3} + Bs^2 + 2Bs + Cs + 2C$$

$$\boxed{\begin{aligned} s^2 + s + 1 &= s^2 + 2\times\frac{s}{2} + \left(\frac{1}{2}\right)^2 + 1 - \left(\frac{1}{2}\right)^2 \\ &= \left(s+\frac{1}{2}\right)^2 + \left(1-\frac{1}{4}\right) \\ &= (s+0.5)^2 + 0.75 \end{aligned}}$$

On equating the coefficient of s² terms, $0 = \dfrac{1}{3} + B$; $\therefore B = -\dfrac{1}{3}$

On equating the coefficient of s terms, $0 = \dfrac{1}{3} + 2B + C$; $\therefore C = -\dfrac{1}{3} - 2B = -\dfrac{1}{3} + \dfrac{2}{3} = \dfrac{1}{3}$

$$T(s) = \frac{\frac{1}{3}}{s} + \frac{-\frac{1}{3}s + \frac{1}{3}}{s^2+s+1} = \frac{1}{3s} - \frac{1}{3}\frac{s}{(s^2+s+1)} + \frac{1}{3}\frac{1}{(s^2+s+1)}$$

$$= \frac{1}{3s} - \frac{1}{3}\frac{s}{(s+0.5)^2+0.75} + \frac{1}{3}\frac{1}{(s+0.5)^2+0.75}$$

2.6   RESPONSE OF FIRST ORDER SYSTEM FOR UNIT STEP INPUT

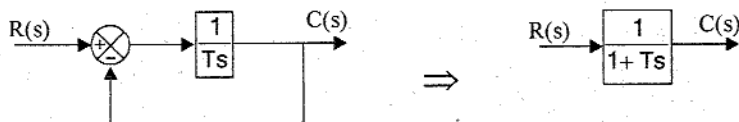The closed loop order system with unity feedback is shown in fig 2.6.

Fig 2.6 : *Closed loop for first order system.*

The closed loop transfer function of first order system, $\dfrac{C(s)}{R(s)} = \dfrac{1}{1+Ts}$

If the input is unit step then, $r(t) = 1$ and $R(s) = \dfrac{1}{s}$.

$\therefore$ The response in s- domain, $C(s) = R(s)\dfrac{1}{(1+Ts)} = \dfrac{1}{s}\dfrac{1}{(1+Ts)} = \dfrac{1}{sT\left(\frac{1}{T}+s\right)} = \dfrac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)}$

By partial fraction expansion,

$$C(s) = \frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)} = \frac{A}{s} + \frac{B}{\left(s+\frac{1}{T}\right)}$$

A is obtained by multiplying C(s) by s and letting s = 0.

$$A = C(s) \times s\big|_{s=0} = \frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)} \times s\bigg|_{s=0} = \frac{\frac{1}{T}}{s+\frac{1}{T}}\bigg|_{s=0} = \frac{\frac{1}{T}}{\frac{1}{T}} = 1$$

B is obtained by multiplying C(s) by (s+1/T) and letting s = −1/T.

$$B = C(s) \times \left(s+\frac{1}{T}\right)\bigg|_{s=-\frac{1}{T}} = \frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)} \times \left(s+\frac{1}{T}\right)\bigg|_{s=-\frac{1}{T}} = \frac{\frac{1}{T}}{s}\bigg|_{s=-\frac{1}{T}} = \frac{\frac{1}{T}}{\frac{-1}{T}} = -1$$

$$\therefore C(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

$$\boxed{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}}$$

The response in time domain is given by,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+\frac{1}{T}}\right\} = 1 - e^{-\frac{t}{T}} \qquad \qquad .....(2.13)$$

The equation (2.13) is the response of the closed loop first order system for unit step input. For step input of step value, A, the equation (2.13) is multiplied by A.

$$\boxed{\begin{array}{l} \therefore \text{For closed loop first order system, Unit step response} = 1 - e^{-\frac{t}{T}} \\[2ex] \qquad\qquad\qquad \text{Step response} \qquad = A\left(1 - e^{-\frac{t}{T}}\right) \end{array}}$$

When, t = 0,   c(t) = $1 - e^0 = 0$
When, t = 1T, c(t) = $1 - e^{-1} = 0.632$
When, t = 2T, c(t) = $1 - e^{-2} = 0.865$
When, t = 3T, c(t) = $1 - e^{-3} = 0.95$
When, t = 4T, c(t) = $1 - e^{-4} = 0.9817$
When, t = 5T, c(t) = $1 - e^{-5} = 0.993$
When, t = ∞,   c(t) = $1 - e^{-\infty} = 1$

Here T is called Time constant of the system. In a time of 5T, the system is assumed to have attained steady state. The input and output signal of the first order system is shown in fig 2.7.
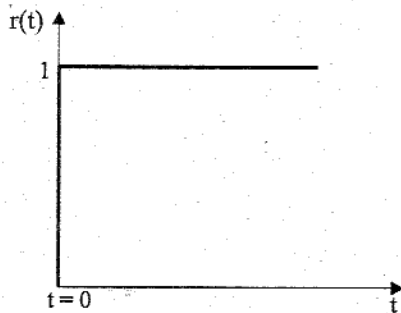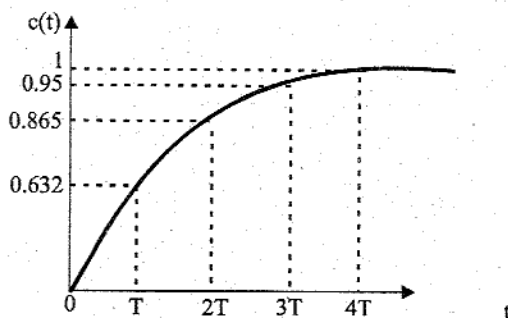
Fig 2.7a : Unit step input.

Fig 2.7b : Response for Unit step input.

Fig 2.7 : Response of first order system to Unit step input.

2.7 SECOND ORDER SYSTEM

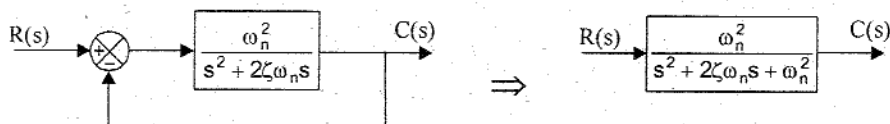The closed loop second order system is shown in fig 2.8

Fig 2.8 : Closed loop for second order system.

The standard form of closed loop transfer function of second order system is given by,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad .....(2.14)$$

where, ω_n = Undamped natural frequency, rad/sec.

ζ = Damping ratio.

*The **damping ratio*** is defined as the ratio of the actual damping to the critical damping. The response c(t) of second order system depends on the value of damping ratio. Depending on the value of ζ, the system can be classified into the following four cases,

Case 1   :   Undamped system, $\zeta = 0$

Case 2   :   Under damped system, $0 < \zeta < 1$

Case 3   :   Critically damped system, $\zeta = 1$

Case 4   :   Over damped system, $\zeta > 1$

The characteristics equation of the second order system is,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \qquad .....(2.15)$$

It is a quadratic equation and the roots of this equation is given by,

$$s_1, s_2 = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = \frac{-2\zeta\omega_n \pm \sqrt{4\omega_n^2(\zeta^2 - 1)}}{2}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \qquad .....(2.16)$$

When $\zeta = 0$, $s_1, s_2 = \pm j\omega_n$; $\begin{cases} \text{roots are purely imaginary} \\ \text{and the system is undamped} \end{cases}$ .....(2.17)

When $\zeta = 1$, $s_1, s_2 = -\omega_n$; $\begin{cases} \text{roots are real and equal and} \\ \text{the system is critically damped} \end{cases}$ .....(2.18)

When $\zeta > 1$, $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$; $\begin{cases} \text{roots are real and unequal and} \\ \text{the system is overdamped} \end{cases}$ .....(2.19)

When $0 < \zeta < 1$, $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm \omega_n\sqrt{(-1)(1-\zeta^2)}$

$$= -\zeta\omega_n \pm \omega_n\sqrt{-1}\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$= -\zeta\omega_n \pm j\omega_d \; ; \begin{cases} \text{roots are complex conjugate} \\ \text{the system is underdamped} \end{cases}$$ .....(2.20)

$$\text{where,} \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$ .....(2.21)

Here ω_d is called damped frequency of oscillation of the system and its unit is rad/sec.

2.7.1 RESPONSE OF UNDAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For undamped system, $z = 0$.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$ .....(2.22)

When the input is unit step, $r(t) = 1$ and $R(s) = \dfrac{1}{s}$.

$\therefore$ The response in s-domain, $C(s) = R(s) \dfrac{\omega_n^2}{s^2 + \omega_n^2} = \dfrac{1}{s} \dfrac{\omega_n^2}{s^2 + \omega_n^2}$ .....(2.23)

By partial fraction expansion,

$$C(s) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)} = \frac{A}{s} + \frac{B}{s^2 + \omega_n^2}$$

A is obtained by multiplying C(s) by s and letting s = 0.

$$A = C(s) \times s\Big|_{s=0} = \frac{\omega_n^2}{s(s^2 + \omega_n^2)} \times s\Big|_{s=0} = \frac{\omega_n^2}{s^2 + \omega_n^2}\Big|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

B is obtained by multiplying C(s) by $(s^2 + \omega_n^2)$ and letting $s^2 = -\omega_n^2$ or $s = j\omega_n$.

$$B = C(s) \times (s^2 + \omega_n^2)\Big|_{s=j\omega} = \frac{\omega_n^2}{s(s^2 + \omega_n^2)} \times (s^2 + \omega_n^2)\Big|_{s=j\omega} = \frac{\omega_n^2}{s}\Big|_{s=j\omega_n} = \frac{\omega_n^2}{j\omega_n} = -j\omega_n = -s$$

$$\therefore C(s) = \frac{A}{s} + \frac{B}{s^2 + \omega_n^2} = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$$

$\mathcal{L}\{1\} = \dfrac{1}{s}$	$\mathcal{L}\{\cos\omega t\} = \dfrac{s}{s^2 + \omega^2}$

Time domain response, $c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s} - \dfrac{s}{s^2 + \omega_n^2}\right\} = 1 - \cos\omega_n t$ .....(2.24)
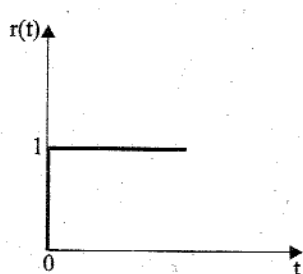
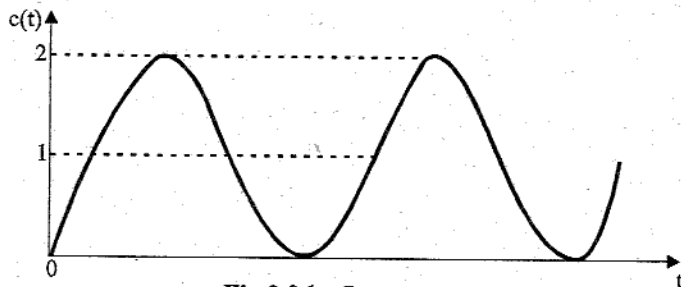Fig 2.9.a : Input.   Fig 2.9.b : Response.

Fig 2.9 : Response of undamped second order system for unit step input.

Using equation (2.24), the response of undamped second order system for unit step input is sketched in fig 2.9, and observed that the response is completely oscillatory.

Note : Every practical system has some amount of damping. Hence undamped system does not exist in practice.

The equation (2.24) is the response of undamped closed loop second order system for unit step input. For step input of step value A, the equation (2.24) should be multiplied by A.

∴ For closed loop undamped second order system,

Unit step response = $1 - \cos \omega_n t$

Step response   = $A(1 - \cos \omega_n t)$

2.7.2   RESPONSE OF UNDERDAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For underdamped system, $0 < \zeta < 1$ and roots of the denominator (characteristic equation) are complex conjugate.

The roots of the denominator are, $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

Since $\zeta < 1$, ζ^2 is also less then 1, and so $1 - \zeta^2$ is always positive.

∴ $s = -\zeta\omega_n \pm \omega_n\sqrt{(-1)(1-\zeta^2)} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

The damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore s = -\zeta\omega_n \pm j\omega_d$$

The response in s-domain, $C(s) = R(s)\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

For unit step input, $r(t) = 1$ and $R(s) = 1/s$.

$$\therefore C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

By partial fraction expansion, $C(s) = \dfrac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \dfrac{A}{s} + \dfrac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ .....(2.25)

A is obtained by multiplying C(s) by s and letting s = 0.

$$\therefore A = s \times C(s)\big|_{s=0} = s \times \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}\bigg|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

To solve for B and C, cross multiply equation (2.25) and equate like power of s.

On cross multiplication equation (2.25) after substituting A = 1, we get,

$$\omega_n^2 = s^2 + 2\zeta\omega_n s + \omega_n^2 + (Bs + C)s$$

$$\omega_n^2 = s^2 + 2\zeta\omega_n s + \omega_n^2 + Bs^2 + Cs$$

Equating coefficients of s^2 we get, $\quad 0 = 1 + B \qquad \therefore B = -1$

Equating coefficient of s we get, $0 = 2\zeta\omega_n + C \qquad \therefore C = -2\zeta\omega_n$

$$\therefore C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad\qquad .....(2.26)$$

Let us add and subtract $\zeta^2\omega_n^2$ to the denominator of second term in the equation (2.26).

$$\therefore C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2 + \zeta^2\omega_n^2 - \zeta^2\omega_n^2} = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2) + (\omega_n^2 - \zeta^2\omega_n^2)}$$

$$= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)} = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \qquad \boxed{\omega_d = \omega_n\sqrt{1 - \zeta^2}}$$

$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \qquad\qquad .....(2.27)$$

Let us multiply and divide by ω_d in the third term of the equation (2.27).

$$\therefore C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d} \frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}}$$

$$\boxed{\mathcal{L}\{e^{-at}\sin\omega t\} = \frac{\omega}{(s + a)^2 + \omega^2}}$$

The response in time domain is given by,

$$\boxed{\mathcal{L}\{e^{-at}\cos\omega t\} = \frac{s + a}{(s + a)^2 + \omega^2}}$$
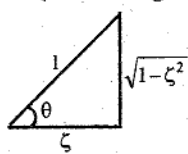
$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d} \frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right\}$$

$$= 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d} e^{-\zeta\omega_n t}\sin\omega_d t = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_n\sqrt{1 - \zeta^2}} \sin\omega_d t\right) \quad \boxed{\omega_d = \omega_n\sqrt{1 - \zeta^2}}$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}}\left(\sqrt{1 - \zeta^2}\,\cos\omega_d t + \zeta\,\sin\omega_d t\right) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}}\left(\sin\omega_d t \times \zeta + \cos\omega_d t \times \sqrt{1 - \zeta^2}\right)$$

Let us express c(t) in a standard form as shown below.

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}}(\sin\omega_d t \times \cos\theta + \cos\omega_d t \times \sin\theta)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}}\sin(\omega_d t + \theta) \qquad .....(2.28)$$

$$\text{where, } \boxed{\theta = \tan^{-1}\frac{\sqrt{1 - \zeta^2}}{\zeta}}$$

> **Note :** On constructing right angle triangle with ζ and $\sqrt{1 - \zeta^2}$, we get
>
> $\sin\theta = \sqrt{1 - \zeta^2}$
>
> $\cos\theta = \zeta$
>
> $\tan\theta = \frac{\sqrt{1 - \zeta^2}}{\zeta}$

The equation (2.28) is the response of under damped closed loop second order system for unit step input. For step input of step value, A, the equation (2.28) should be multiplied by A.

$\therefore$ For closed loop under damped second order system,

$$\text{Unit step response} = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta) \; ; \;\; \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\text{Step response} = A\left[1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta) \right] \; ; \;\; \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

Using equation (2.28) the response of underdamped second order system for unit step input is sketched and observed that the response oscillates before settling to a final value. The oscillations depends on the value of damping ratio.
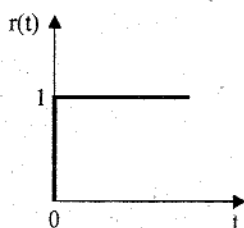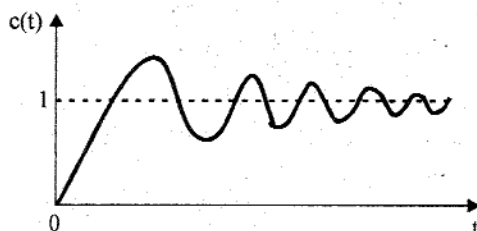

Fig 2.10.a : Input.   Fig 2.10.b : Response.

Fig 2.10 : Response of under damped second order system for unit step input.

2.7.3 RESPONSE OF CRITICALLY DAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For critical damping $\zeta = 1$.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\omega_n)^2} \quad\quad\quad\quad .....(2.29)$$

When input is unit step, $r(t) = 1$ and $R(s) = 1/s$.

$\therefore$ The response in s-domain,

$$C(s) = R(s)\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{1}{s}\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{\omega_n^2}{s(s+\omega_n)^2} \quad\quad .....(2.30)$$

By partial fraction expansion, we can write,

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n}$$

$$A = s \times C(s)\Big|_{s=0} = \frac{\omega_n^2}{(s+\omega_n)^2}\Big|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = (s+\omega_n)^2 \times C(s)\Big|_{s=-\omega_n} = \frac{\omega_n^2}{s}\Big|_{s=-\omega_n} = -\omega_n$$

$$C = \frac{d}{ds}\Big[(s+\omega_n)^2 \times C(s)\Big]\Big|_{s=-\omega_n} = \frac{d}{ds}\left(\frac{\omega_n^2}{s}\right)\Big|_{s=-\omega_n} = \frac{-\omega_n^2}{s^2}\Big|_{s=-\omega_n} = -1$$

$$\therefore C(s) = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n} = \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}$$

The response in time domain,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}\right\}$$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}}$$

$$\boxed{\mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}}$$

$$c(t) = 1 - \omega_n t\, e^{-\omega_n t} - e^{-\omega_n t}$$

$$\boxed{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}}$$

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t) \qquad\qquad\qquad .....(2.31)$$

The equation (2.31) is the response of critically damped closed loop second order system for unit step input. For step input of step value, A, the equation (2.31) should be multiplied by A.

$$\boxed{\begin{array}{l} \therefore \text{ For closed loop critically damped second order system,} \\[4pt] \qquad \text{Unit step response} = 1 - e^{-\omega_n t}(1 + \omega_n t) \\[4pt] \qquad \text{Step response} \qquad = A\left[1 - e^{-\omega_n t}(1 + \omega_n t)\right] \end{array}}$$

Using equation (2.31), the response of critically damped second order system is sketched as shown in fig 2.11 and observed that the response has no oscillations.
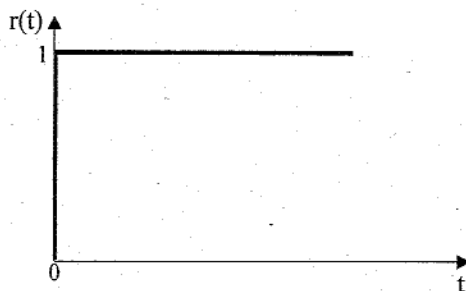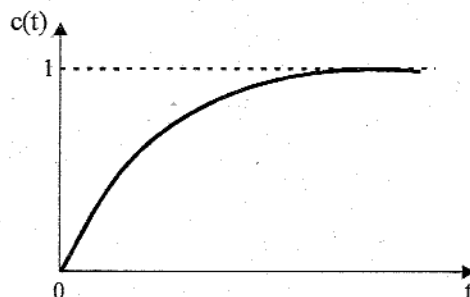

Fig 2.11.a : Input.   Fig 2.11.b : Response.

Fig 2.11 : Response of critically damped second order system for unit step input.

2.7.4   RESPONSE OF OVER DAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For overdamped system $\zeta > 1$. The roots of the denominator of transfer function are real and distinct. Let the roots of the denominator be s_a, s_b.

$$s_a,\, s_b = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\left[\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}\right] \qquad .....(2.32)$$

Let $s_1 = -s_2$ and $s_2 = -s_b$ $\qquad \therefore s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \qquad .....(2.33)$

$$s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \qquad .....(2.34)$$

The closed loop transfer function can be written in terms of s_1 and s_2 as shown below.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+s_1)(s+s_2)} \qquad .....(2.35)$$

For unit step input $r(t) = 1$ and $R(s) = 1/s$.

$$\therefore C(s) = R(s)\frac{\omega_n^2}{(s+s_1)(s+s_2)} = \frac{\omega_n^2}{s(s+s_1)(s+s_2)}$$

By partial fraction expansion we can write,

$$C(s) = \frac{\omega_n^2}{s(s+s_1)(s+s_2)} = \frac{A}{s} + \frac{B}{s+s_1} + \frac{C}{s+s_2}$$

$$A = s \times C(s)|_{s=0} = s \times \frac{\omega_n^2}{s(s+s_1)(s+s_2)}\bigg|_{s=0} = \frac{\omega_n^2}{s_1 s_2}$$

$$= \frac{\omega_n^2}{\left[\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]\left[\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]} = \frac{\omega_n^2}{\zeta^2\omega_n^2 - \omega_n^2(\zeta^2-1)} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = (s+s_1) \times C(s)|_{s=-s_1} = \frac{\omega_n^2}{s(s+s_2)}\bigg|_{s=-s_1} = \frac{\omega_n^2}{-s_1(-s_1+s_2)}$$

$$= \frac{-\omega_n^2}{s_1\left[-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]} = \frac{-\omega_n^2}{\left[2\omega_n\sqrt{\zeta^2-1}\right]s_1} = \frac{-\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{s_1}$$

$$C = C(s) \times (s+s_2)|_{s=-s_2} = \frac{\omega_n^2}{s(s+s_1)}\bigg|_{s=-s_2} = \frac{\omega_n^2}{-s_2(-s_2+s_1)}$$

$$= \frac{\omega_n^2}{-s_2\left[-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]} = \frac{\omega_n^2}{\left[2\omega_n\sqrt{\zeta^2-1}\right]s_2} = \frac{\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{s_2}$$

The response in time domain, $c(t)$ is given by,

$$c(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{s_1}\frac{1}{(s+s_1)} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{s_2}\frac{1}{(s+s_2)}\right\}$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{s_1}e^{-s_1 t} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{s_2}e^{-s_2 t}$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right) \qquad\qquad\qquad .....(2.36)$$

where, $s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$

$$s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$$

The equation (2.36) is the response of overdamped closed loop system for unit step input. For step input of value, A, the equation (2.36) is multiplied by A.

$\therefore$ For closed loop over damped second order system,

Unit step response $= 1 - \dfrac{\omega_n}{2\sqrt{\zeta^2-1}}\dfrac{1}{s_1}\left(\dfrac{e^{-s_1 t}}{s_1} - \dfrac{e^{-s_2 t}}{s_2}\right)$ where, $s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$

Step response $= A\left[1 - \dfrac{\omega_n}{2\sqrt{\zeta^2-1}}\dfrac{1}{s_1}\left(\dfrac{e^{-s_1 t}}{s_1} - \dfrac{e^{-s_2 t}}{s_2}\right)\right]$ $s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$
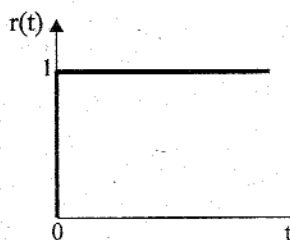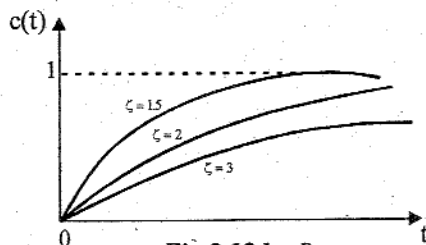
Fig 2.12.a : Input.

Fig 2.12.b : Response.

Fig 2.12 : Response of over damped second order system for unit step input.

Using equation (2.36), the response of overdamped second order system is sketched as shown in fig 2.12 and observed that the response has no oscillations but it takes longer time for the response to reach the final steady value.

2.8 TIME DOMAIN SPECIFICATIONS

The desired performance characteristics of control systems are specified in terms of time domain specifications. Systems with energy storage elements cannot respond instantaneously and will exhibit transient responses, whenever they are subjected to inputs or disturbances.

The desired performance characteristics of a system of any order may be specified in terms of the transient response to a unit step input signal. The response of a second order system for unit-step input with various values of damping ratio is shown in fig 2.13.
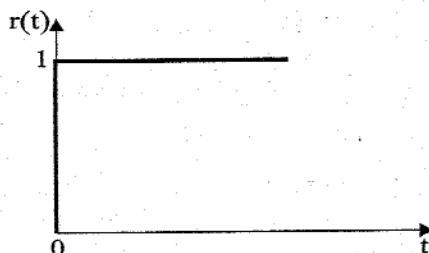
Fig 2.13.a : Input.
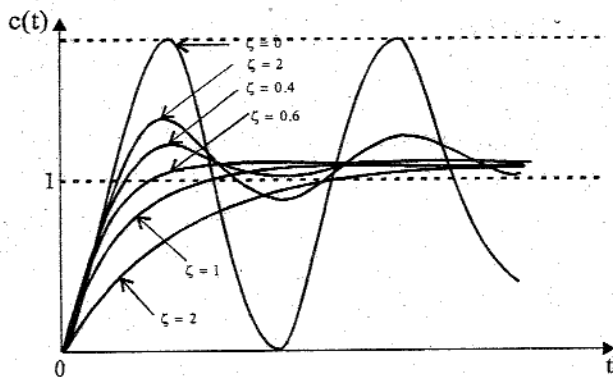
Fig 2.13.b : Response.

Fig 2.13 : Unit step response of second order system.

The transient response of a system to a unit step input depends on the initial conditions. Therefore to compare the time response of various systems it is necessary to start with standard initial conditions. The most practical standard is to start with the system at rest and so output and all time derivatives before $t = 0$ will be zero. The transient response of a practical control system often exhibits damped oscillation before reaching steady state. A typical damped oscillatory response of a system is shown in fig 2.14.

The transient response characteristics of a control system to a unit step input is specified in terms of the following time domain specifications.

1. Delay time, t_d
2. Rise time, t_r
3. Peak time, t_p
4. Maximum overshoot, M_p
5. Settling time, t_s
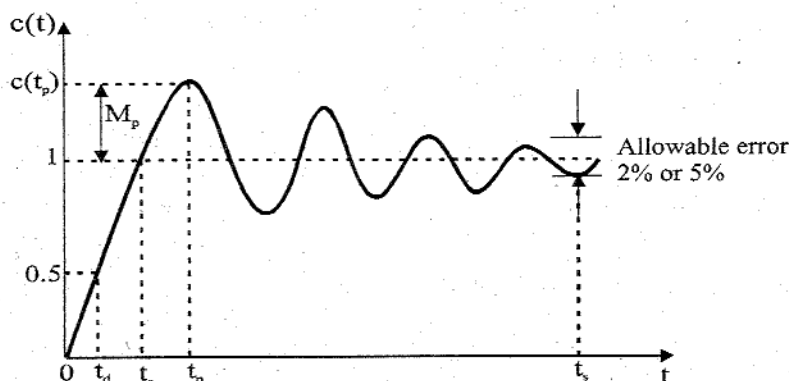
Fig 2.14 : *Damped oscillatory response of second order system for unit step input.*

The time domain specifications are defined as follows.

1. DELAY TIME (t_d) : It is the time taken for response to reach 50% of the final value, for the very first time.

2. RISE TIME (t_r) : It is the time taken for response to raise from 0 to 100% for the very first time. For underdamped system, the rise time is calculated from 0 to 100%. But for overdamped system it is the time taken by the response to raise from 10% to 90%. For critically damped system, it is the time taken for response to raise from 5% to 95%.

3. PEAK TIME (t_p) : It is the time taken for the response to reach the peak value the very first time. (or) It is the time taken for the response to reach the peak overshoot, M_p.

4. PEAK OVERSHOOT (M_p) : It is defined as the ratio of the maximum peak value to the final value, where the maximum peak value is measured from final value.

Let, $c(\infty)$ = Final value of c(t).

$c(t_p)$ = Maximum value of c(t).

Now, Peak overshoot, $M_p = \dfrac{c(t_p) - c(\infty)}{c(\infty)}$ .....(2.37)

% Peak overshoot, $\%M_p = \dfrac{c(t_p) - c(\infty)}{c(\infty)} \times 100$ .....(2.38)

5. SETTLING TIME (t_s) : It is defined as the time taken by the response to reach and stay within a specified error. It is usually expressed as % of final value. The usual tolerable error is 2 % or 5% of the final value.

EXPRESSIONS FOR TIME DOMAIN SPECIFICATIONS

Rise time (t_r)

The unit step response of second order system for underdamped case is given by,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

At $t = t_r$, $c(t) = c(t_r) = 1$ **(Refer fig 2.14).**

3. *If any coefficient a_i is negative then atleast one root is in the right half of s- plane.*

It can be concluded that the absence or negativeness of any of the coefficients of a characteristic polynomial indicates that the system is either unstable or at most marginally stable. Thus *the necessary condition for stability of the system is that all the coefficients of its characteristic polynomial be positive.* If any coefficient is zero/negative, we can immediately say that the system is unstable.

In order for all the roots to have negative real parts, it is necessary that all of the coefficients of characteristic equation be positive, but it is not sufficient, because there may be roots in the right half plane and/or on the imaginary axis, even when coefficients are positive.(i.e., when roots have negative real part, then all the coefficients of characteristic polynomial will be positive, but the reverse condition is not true always).

<div align="center">

always true

Roots with negative real part.   All coefficients positive.

not always true

</div>
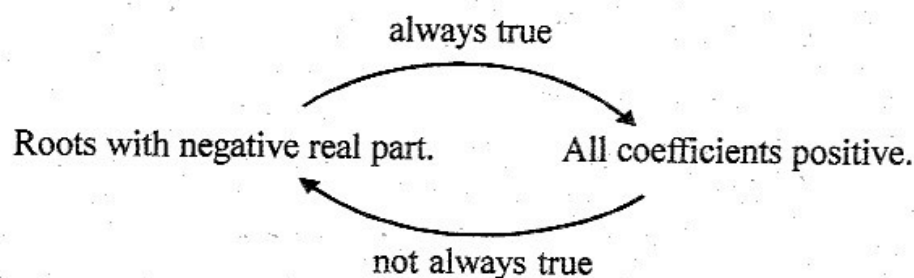
Hence, when all the coefficients are positive, the system may or may not be stable, because there may be roots in the right half plane and/or on the imaginary axis.

For example, consider the characteristic polynomial with all positive coefficients,

$$s^3 + s^2 + 2s + 8 = 0.$$

The characteristic polynomial can be written as,

$$(s^3 + s^2 + 2s + 8) = (s+2)\left(s - \frac{1}{2} - j\frac{\sqrt{15}}{2}\right)\left(s - \frac{1}{2} + j\frac{\sqrt{15}}{2}\right) = 0$$

Now the roots are,

$$s = -2, \quad +\frac{1}{2} + j\frac{\sqrt{15}}{2}, \quad +\frac{1}{2} - j\frac{\sqrt{15}}{2}$$

The coefficients of the polynomial are all positive, but two roots have positive real part and so will lie on on right half of s-plane, therefore the system is unstable.

4.3 ROUTH HURWITZ CRITERION

The Routh-Hurwitz stability criterion is an analytical procedure for determining whether all the roots of a polynomial have negative real part or not.

The first step in analysing the stability of a system is to examine its characteristic equation. The necessary condition for stability is that all the coefficients of the polynomial be positive. If some of the coefficients are zero or negative it can be concluded that the system is not stable.

When all the coefficients are positive, the system is not necessarily stable. Eventhough the coefficient are positive, some of the roots may lie on the right half of s-plane or on the imaginary axis. In order for all the roots to have negative real parts, it is necessary but not sufficient that all coefficients of the characteristic equation be positive. If all the coefficients of the characteristic equation are positive, then the system may be stable and one should proceed further to examine the sufficient conditions of stability.

A. Hurwitz and E.J. Routh independently published the method of investigating the sufficient conditions of stability of a system. The Hurwitz criterion is in terms of determinants and Routh criterion is in terms of array formulation. The Routh stability criterion is presented here.

The Routh stability criterion is based on ordering the coefficients of the characteristic equation, into a schedule, called the Routh array as shown below.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + ... + a_{n-1} s + a_n = 0, \text{ where } a_0 > 0,$$

s^n :	a_0	a_2	a_4	a_6	a_8	......
s^{n-1} :	a_1	a_3	a_5	a_7	a_9	......
s^{n-2} :	b_0	b_1	b_2	b_3	b_4	......
s^{n-3} :	c_0	c_1	c_2	c_3	c_4	......
s^1 :	g_0					
s_0 :	h_0					

The Routh stability criterion can be stated as follows.

"The necessary and sufficient condition for stability is that all of the elements in the first column of the Routh array be positive. If this condition is not met, the system is unstable and the number of sign changes in the elements of the first column of the Routh array corresponds to the number of roots of the characteristic equation in the right half of the s-plane".

> Note : If the order of sign of first column element is +, +, −, + and +. Then + to − is considered as one sign change and − to + as another sign change.

CONSTRUCTION OF ROUTH ARRAY

Let the characteristic polynomial be,

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_3 s^{n-3} + ... a_{n-1} s^1 + a_n s^0$$

The coefficients of the polynomial are arranged in two rows as shown below.

s^n :	a_0	a_2	a_4	a_6	.....
s^{n-1} :	a_1	a_3	a_5	a_7	.....

When n is even, the s^n row is formed by coefficients of even order terms (i.e., coefficients of even powers of s) and s^{n-1} row is formed by coefficients of odd order terms (i.e., coefficients of odd powers of s).

When n is odd, the s^n row is formed by coefficients of odd order terms (i.e., coefficients of odd powers of s) and s^{n-1} row is formed by coefficients of even order terms(i.e., coefficients of even powers of s) .

The other rows of routh array upto s^0 row can be formed by the following procedure. Each row of Routh array is constructed by using the elements of previous two rows.

Consider two consecutive rows of Routh array as shown below.

$$s^{n-x} \quad : \quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 .....$$
$$s^{n-x-1} : \quad y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5 .....$$

Let the next row be,

$$s^{n-x-2} : \quad z_0 \quad z_1 \quad z_2 \quad z_3 \quad z_4 .....$$

The elements of s^{n-x-2} row are given by,

$$z_0 = \frac{(-1)\begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix}}{y_0} = \frac{y_0 x_1 - y_1 x_0}{y_0}$$

$$z_1 = \frac{(-1)\begin{vmatrix} x_0 & x_2 \\ y_0 & y_2 \end{vmatrix}}{y_0} = \frac{y_0 x_2 - y_2 x_0}{y_0}$$

$$z_2 = \frac{(-1)\begin{vmatrix} x_0 & x_3 \\ y_0 & y_3 \end{vmatrix}}{y_0} = \frac{y_0 x_3 - y_3 x_0}{y_0}$$

$$z_3 = \frac{(-1)\begin{vmatrix} x_0 & x_4 \\ y_0 & y_4 \end{vmatrix}}{y_0} = \frac{y_0 x_4 - y_4 x_0}{y_0}$$

$$z_4 = \frac{(-1)\begin{vmatrix} x_0 & x_5 \\ y_0 & y_5 \end{vmatrix}}{y_0} = \frac{y_0 x_5 - y_5 x_0}{y_0} \quad \text{and so on.}$$

The elements $z_0, z_1, z_2, z_3, ...$ are computed for all possible computations as shown above.

In the process of constructing Routh array the missing terms are considered as zeros. Also, all the elements of any row can be multiplied or divided by a positive constant to simplify the computational work.

In the construction of Routh array one may come across the following three cases.

Case-I : Normal Routh array (Non-zero elements in the first column of routh array).

Case-II : A row of all zeros.

Case-III : First element of a row is zero but some or other elements are not zero.

Case-I : Normal routh array

In this case, there is no difficulty in forming routh array. The routh array can be constructed as explained above. The sign changes are noted to find the number of roots lying on the right half of s-plane and the stability of the system can be estimated.

In this case,

1. If there is no sign change in the first column of Routh array then all the roots are lying on left half of s-plane and the system is stable.

2. If there is sign change in the first column of routh array, then the system is unstable and the number of roots lying on the right half of s-plane is equal to number of sign changes. The remaining roots are lying on the left half of s-plane.

Case-II : A row of all zeros

An all zero row indicates the existence of an even polynomial as a factor of the given characteristic equation. In an even polynomial the exponents of s are even integers or zero only. This even polynomial factor is also called **auxiliary polynomial.** The coefficients of the auxiliary polynomial will always be the elements of the row directly above the row of zeros in the array.

The roots of an even polynomial occur in pairs that are equal in magnitude and opposite in sign. Hence, these roots can be purely imaginary, purely real or complex. The purely imaginary and purely real roots occur in pairs. The complex roots occur in groups of four and the complex roots have quadrantal symmetry, that is the roots are symmetrical with respect to both the real and imaginary axes. The fig 4.1 shows the roots of an even polynomial.
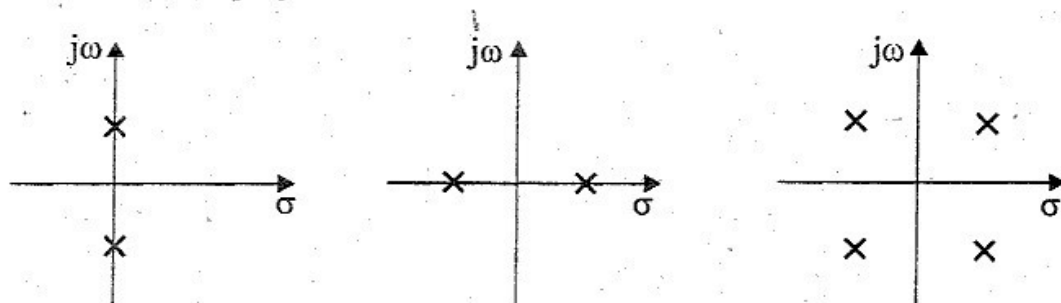
Fig 4.1 : *The roots of an even polynomial.*

The case-II polynomial can be analyzed by any one of the following two methods.

METHOD-1

1. Determine the auxiliary polynomial, A(s)
2. Differentiate the auxiliary polynomial with respect to s, to get d A(s)/ds
3. The row of zeros is replaced with coefficients of dA(s)/ds.
4. Continue the construction of the array in the usual manner (as that of case-I) and the array is interpreted as follows.

    a. If there are sign changes in the first column of routh array then the system is unstable. The number of roots lying on right half of s-plane is equal to number of sign changes. The number of roots on imaginary axis can be estimated from the roots of auxiliary polynomial. The remaining roots are lying on the left half of s-plane.

    b. If there are no sign changes in the first column of routh array then the all zeros row indicate the existence of purely imaginary roots and so the system is limitedly or marginally stable. The roots of auxiliary equation lies on imaginary axis and the remaining roots lies on left half of s-plane.

METHOD-2

1. Determine the auxiliary polynomial, A(s).
2. Divide the characteristic equation by auxiliary polynomial.

3.   Construct Routh array using the coefficients of quotient polynomial.

4.   The array is interpretted as follows.

   a.   If there are sign changes in the first column of routh array of quotient polynomial then the system is unstable. The number of roots of quotient polynomial lying on right half of s-plane is given by number of sign changes in first column of routh array.

   The roots of auxiliary polynomial are directly calculated to find whether they are purely imaginary or purely real or complex.

   The total number of roots on right half of s-plane is given by the sum of number of sign changes and the number of roots of auxiliary polynomial with positive real part. The number of roots on imaginary axis can be estimated from the roots of auxiliary polynomial. The remaining roots are lying on the left half of s-plane.

   b.   If there is no sign change in the first column of routh array of quotient polynomial then the system is limitedly or marginally stable. Since there is no sign change all the roots of quotient polynomial are lying on the left half of s-plane.

   The roots of auxiliary polynomial are directly calculated to find whether they are purely imaginary or purely real or complex. The number of roots lying on imaginary axis and on the right half of s-plane can be estimated from the roots of auxiliary polynomial. The remaining roots are lying on the left half of s-plane.

Case-III : First element of a row is zero

While constructing routh array, if a zero is encountered as first element of a row then all the elements of the next row will be infinite. To overcome this problem let $0 \to \in$ and complete the construction of array in the usual way (as that of case-I)

Finally let $\in \to 0$ and determine the values of the elements of the array which are functions of $\in$. The resultant array is interpreted as follows.

Note : *If all the elements of a row are zeros then the solution is attempted by considering the polynomial as case-II polynomial. Even if there is a single element zero on s^1 row, it is considered as a row of all zeros.*

---

   a.   If there is no sign change in first column of routh array and if there is no row with all zeros, then all the roots are lying on left half of s-plane and the system is stable.

   b.   If there are sign changes in first column of routh array and there is no row with all zeros, then some of the roots are lying on the right half of s-plane and the system is unstable. The number of roots lying on the right half of s-plane is equal to number of sign changes and the remaining roots are lying on the left half of s-plane.

   c.   If there is a row of all zeros after letting $\in \to 0$, then there is a possibility of roots on imaginary axis. Determine the auxiliary polynomial and divide the characteristic equation by auxiliary polynomial to eliminate the imaginary roots. The routh array is constructed using the coefficients of quotient polynomial and the characteristic equation is interpreted as explained in method-2 of case-II polynomial.

EXAMPLE 4.1

Using Routh criterion, determine the stability of the system represented by the characteristic equation, $s^4 + 8s^3 + 18s^2 + 16s + 5 = 0$. Comment on the location of the roots of characteristic equation.

SOLUTION

The characteristic equation of the system is, $s^4 + 8s^3 + 18s^2 + 16s + 5 = 0$.

The given characteristic equation is 4th order equation and so it has 4 roots. Since the highest power of s is even number, form the first row of routh array using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

$$
\begin{array}{llll}
s^4 & : \quad 1 & 18 & 5 \qquad \text{.... Row-1} \\
s^3 & : \quad 8 & 16 & \qquad \text{.... Row-2}
\end{array}
$$

The elements of s^3 row can be divided by 8 to simplify the computations.

$$
\begin{array}{llll}
s^4 & : \quad 1 & 18 & 5 \qquad \text{.... Row-1} \\
s^3 & : \quad 1 & 2 & \qquad \text{.... Row-2} \\
s^2 & : \quad 16 & 5 & \qquad \text{.... Row-3} \\
s^1 & : \quad 1.7 & & \qquad \text{.... Row-4} \\
s^0 & : \quad 5 & & \qquad \text{.... Row-5}
\end{array}
$$

Column-1

$$
\begin{aligned}
s^2 &: \quad \frac{1 \times 18 - 2 \times 1}{1} \quad \frac{1 \times 5 - 0 \times 1}{1} \\
s^2 &: \quad 16 \qquad\qquad 5
\end{aligned}
$$

$$
\begin{aligned}
s^1 &: \quad \frac{16 \times 2 - 5 \times 1}{16} \\
s^1 &: \quad 1.6875 \approx 1.7
\end{aligned}
$$

$$
\begin{aligned}
s^0 &: \quad \frac{1.7 \times 5 - 0 \times 16}{17} \\
s^0 &: \quad 5
\end{aligned}
$$

On examining the elements of first column of routh array it is observed that all the elements are positive and there is no sign change. Hence all the roots are lying on the left half of s-plane and the system is stable.

RESULT

1. Stable system

2. All the four roots are lying on the left half of s-plane.

EXAMPLE 4.2

Construct Routh array and determine the stability of the system whose characterisitc equation is $s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$. Also determine the number of roots lying on right half of s-plane, left half of s-plane and on imaginary axis.

SOLUTION

The characteristic equation of the system is, $s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$.

The given characteristic polynomial is 6th order equation and so it has 6 roots. Since the highest power of s is even number, form the first row of routh array using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

$$
\begin{array}{lllll}
s^6 & : \quad 1 & 8 & 20 & 16 \qquad \text{.... Row-1} \\
s^5 & : \quad 2 & 12 & 16 & \qquad \text{.... Row-2}
\end{array}
$$

The elements of s^5 row can be divided by 2 to simplify the calculations.

$$s^6 \quad : \quad \boxed{1} \quad 8 \quad 20 \quad 16 \quad \text{.... Row-1}$$

$$s^5 \quad : \quad 1 \quad 6 \quad 8 \quad \text{....Row-2}$$

$$s^4 \quad : \quad 1 \quad 6 \quad 8 \quad \text{....Row-4}$$

$$s^3 \quad : \quad 0 \quad 0 \quad \text{....Row-4}$$

$$s^3 \quad : \quad 1 \quad 3 \quad \text{....Row-4}$$

$$s^2 \quad : \quad 3 \quad 8 \quad \text{....Row-5}$$

$$s^1 \quad : \quad 0.33 \quad \text{....Row-6}$$

$$s^0 \quad : \quad 8 \quad \text{....Row-7}$$

↑ Column-1

$$s^4 \quad : \quad \frac{1 \times 8 - 6 \times 1}{1} \quad \frac{1 \times 20 - 8 \times 1}{1} \quad \frac{1 \times 16 - 0 \times 1}{1}$$

$$s^4 \quad : \quad 2 \quad 12 \quad 16$$

divide by 2

$$s^4 \quad : \quad 1 \quad 6 \quad 8$$

$$s^3 \quad : \quad \frac{1 \times 6 - 6 \times 1}{1} \quad \frac{1 \times 8 - 8 \times 1}{1}$$

$$s^3 \quad : \quad 0 \quad 0$$

The auxiliary equation is, $A = s^4 + 6s^2 + 8$. On differentiating A with respect to s we get,

$$\frac{dA}{ds} = 4s^3 + 12s$$

The coefficients of $\dfrac{dA}{ds}$ are used to form s^3 row.

$$s^3 : \quad 4 \quad 12$$

divide by 4

$$s^3 : \quad 1 \quad 3$$

$$s^2 : \quad \frac{1 \times 6 - 3 \times 1}{1} \quad \frac{1 \times 8 - 0 \times 1}{1}$$

$$s^2 : \quad 3 \quad 8$$

$$s^1 : \quad \frac{3 \times 3 - 8 \times 1}{3}$$

$$s^1 : \quad 0.33$$

$$s^0 : \quad \frac{0.33 \times 8 - 0 \times 3}{0.33}$$

$$s^0 : \quad 8$$

On examining the elements of 1st column of routh array it is observed that there is no sign change. The row with all zeros indicate the possibility of roots on imaginary axis. Hence the system is limitedly or marginally stable.

The auxiliary polynomial is,

$$s^4 + 6s^2 + 8 = 0$$

Let, $s^2 = x$

$$\therefore x^2 + 6x + 8 = 0$$

The roots of quadratic are, $\quad x = \dfrac{-6 \pm \sqrt{6^2 - 4 \times 8}}{2}$

$$= -3 \pm 1 = -2 \text{ or } -4$$

The roots of auxiliary polynomial is,

$$s = \pm\sqrt{x} = \pm\sqrt{-2} \text{ and } \pm\sqrt{-4}$$

$$= +j\sqrt{2}, -j\sqrt{2}, +j2 \text{ and } -j2$$

The roots of auxiliary polynomial are also roots of characteristic equation. Hence 4 roots are lying on imaginary axis and the remaining two roots are lying on the left half of s-plane.

RESULT

1. The system is limitedly or marginally stable.

2. Four roots are lying on imaginary axis and remaining two roots are lying on left half of s-plane.

EXAMPLE 4.3

Construct Routh array and determine the stability of the system represented by the characteristic equation, $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$. Comment on the location of the roots of characteristic equation.

SOLUTION

The characteristic equation of the system is, $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$.

The given characteristic polynomial is 5th order equation and so it has 5 roots. Since the highest power of s is odd number, form the first row of routh array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

$$s^5 \quad : \quad 1 \quad 2 \quad 3 \quad \text{.... Row-1}$$

$$s^4 \quad : \quad 1 \quad 2 \quad 5 \quad \text{.... Row-2}$$

$$s^3 : \quad \epsilon \qquad\qquad -2 \qquad \dots \text{Row-3}$$

$$s^2 : \quad \frac{2\,\epsilon + 2}{\epsilon} \qquad\qquad 5 \qquad \dots \text{Row-4}$$

$$s^1 : \quad \frac{-\left(5\,\epsilon^2 + 4\,\epsilon + 4\right)}{2\,\epsilon + 2} \qquad \dots \text{Row-5}$$

$$s^0 : \quad 5 \qquad\qquad \dots \text{Row-6}$$

On letting $\epsilon \to 0$, we get

		Column-1			
s^5	:	1	2	3	.... Row-1
s^4	:	1	2	5	.... Row-2
s^3	:	0	-2		.... Row-3
s^2	:	∞	5		.... Row-4
s^1	:	-2			.... Row-5
s^0	:	5			.... Row-6

On the right side:

$$s^3 : \frac{1 \times 2 - 2 \times 1}{1} \qquad \frac{1 \times 3 - 5 \times 1}{1}$$

$$s^3 : \quad 0 \qquad\qquad -2$$

Replace 0 by ϵ

$$s^3 : \quad \epsilon \qquad\qquad -2$$

$$s^2 : \frac{\epsilon \times 2 - (-2 \times 1)}{\epsilon} \qquad \frac{\epsilon \times 5 - 0 \times 1}{\epsilon}$$

$$s^2 : \frac{2\,\epsilon + 2}{\epsilon} \qquad\qquad 5$$

$$s^1 : \frac{\dfrac{2\,\epsilon + 2}{\epsilon} \times (-2) - (5 \times \epsilon)}{\dfrac{2\,\epsilon + 2}{\epsilon}}$$

$$s^1 : \frac{-\left(5\,\epsilon^2 + 4\,\epsilon + 4\right)}{2\,\epsilon + 2}$$

$$s^0 : \frac{\dfrac{-\left(5\,\epsilon^2 + 4\,\epsilon + 4\right)}{2\,\epsilon + 2} \times 5 - 0 \times \dfrac{2\,\epsilon + 2}{\epsilon}}{\dfrac{-\left(5\,\epsilon^2 + 4\,\epsilon + 4\right)}{2\,\epsilon + 2}}$$

$$s^0 : 5$$

On observing the elements of first column of routh array, it is found that there are two sign changes. Hence two roots are lying on the right half of s-plane and the system is unstable. The remaining three roots are lying on the left half of s-plane.

RESULT

(a). The system is unstable.

(b). Two roots are lying on right half of s-plane and three roots are lying on left half of s-plane.

EXAMPLE 4.4

By routh stability criterion determine the stability of the system represented by the characteristic equation, $9s^5 - 20s^4 + 10s^3 - s^2 - 9s - 10 = 0$. Comment on the location of roots of characteristic equation.

SOLUTION

The characteristic polynomial of the system is, $9s^5 - 20s^4 + 10s^3 - s^2 - 9s - 10 = 0$

On examining the coefficients of the characteristic polynomial, it is found that some of the coefficients are negative and so some roots will lie on the right half of s-plane. Hence the system is unstable. The routh array can be constructed to find the number of roots lying on right half of s-plane.

The given characteristic polynomial is 5th order equation and so it has 5 roots. Since the highest power of s is odd number, form the first row of routh array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

		Column-1			
s^5	:	9	10	-9	.... Row-1
s^4	:	-20	-1	-10	.... Row-2
s^3	:	9.55	-13.5		.... Row-3
s^2	:	-29.3	-10		.... Row-4
s^1	:	-16.8			.... Row-5
s^0	:	-10			.... Row-6

$$s^3 : \frac{-20 \times 10 - (-1) \times 9}{-20} \qquad \frac{-20 \times (-9) - (-10) \times 9}{-20}$$

$$s^3 : 9.55 \qquad -13.5$$

$$s^2 : \frac{9.55 \times (-1) - (-13.5) \times (-20)}{9.55} \qquad \frac{9.55 \times (-10)}{9.55}$$

$$s^2 : -29.3 \qquad -10$$

By examining the elements of Ist column of routh array it is observed that there are three sign changes and so three roots are lying on the right half of s-plane and the remaining two roots are lying on the left half of s-plane.

$$s^1: \frac{-29.3 \times (-13.5) - (-10) \times 9.55}{-29.3}$$

$$s^1: -16.8$$

RESULT

(a). The system is unstable.

$$s^0: \frac{-16.8 \times (-10)}{-16.8}$$

$$s^0: -10$$

(b). Three roots are lying on right half of s-plane and two roots are lying on left half of s-plane.

EXAMPLE 4.5

The characteristic polynomial of a system is, $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$. Determine the location of roots on s-plane and hence the stability of the system.

SOLUTION

METHOD-I

The characteristic equation is, $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$.

The given characteristic polynomial is 7th order equation and so it has 7 roots. Since the highest power of s is odd number, form the first row of array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s as shown below.

s^7 :	1	24	24	23	.... Row-1
s^6 :	9	24	24	15	.... Row-2

Divide s^6 row by 3 to simplify the computations.

s^7 :	1	24	24	23	.... Row-1
s^6 :	3	8	8	5	.... Row-2
s^5 :	1	1	1		.... Row-3
s^4 :	1	1	1		.... Row-4
s^3 :	0	0			.... Row-5
s^3 :	2	1			.... Row-5
s^2 :	0.5	1			.... Row-6
s^1 :	-3				.... Row-7
s^0 :	1				.... Row-8

Column-1

$$s^5: \frac{3 \times 24 - 8 \times 1}{3} \quad \frac{3 \times 24 - 8 \times 1}{3} \quad \frac{3 \times 23 - 5 \times 1}{3}$$

$$s^5: 21.33 \qquad 21.33 \qquad 21.33$$

Divide by 21.33

$$s^5: 1 \qquad\qquad 1 \qquad\qquad 1$$

$$s^4: \frac{1 \times 8 - 1 \times 3}{1} \quad \frac{1 \times 8 - 1 \times 3}{1} \quad \frac{1 \times 5 - 0 \times 3}{1}$$

$$s^4: 5 \qquad\qquad 5 \qquad\qquad 5$$

Divide by 5

$$s^4: 1 \qquad\qquad 1 \qquad\qquad 1$$

$$s^3: \frac{1 \times 1 - 1 \times 1}{1} \quad \frac{1 \times 1 - 1 \times 1}{1}$$

$$s^3: 0 \qquad\qquad 0$$
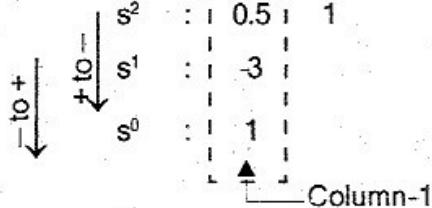
On examining the first column elements of routh array it is found that there are two sign changes. Hence two roots are lying on the right half of s-plane and so the system is unstable.

The row of all zeros indicates the possibility of roots on imaginary axis. This can be tested by evaluating the roots of auxiliary polynomial.

The auxiliary equation is, $s^4 + s^2 + 1 = 0$

Put, $s^2 = x$ in the auxiliary equation,

$$s^4 + s^2 + 1 = x^2 + x + 1 = 0$$

The auxiliary polynomial is,

$$A = s^4 + s^2 + 1$$

Differentiate A with respect to s.

$$\frac{dA}{ds} = 4s^3 + 2s$$

$$s^3: 4 \quad 2$$

Divide by 2

$$s^3: 2 \quad 1$$

The roots of quadratic are, $x = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\dfrac{1}{2} \pm j\dfrac{\sqrt{3}}{2}$

$\qquad = 1\angle 120°$ or $1\angle -120°$

But $s^2 = x$, $\therefore s = \pm\sqrt{x} = \pm\sqrt{1\angle 120°}$ $\qquad$ or $\qquad \pm\sqrt{1\angle -120°}$

$\qquad\qquad = \pm\sqrt{1}\angle 120°/2 \qquad$ or $\qquad \pm\sqrt{1}\angle -120°/2$

$\qquad\qquad = \pm 1\angle 60° \qquad\qquad$ or $\qquad \pm 1\angle -60°$

$\qquad\qquad = \pm(0.5 + j0.866) \quad$ or $\quad \pm(0.5 - j0.866)$

s^2 :	$\dfrac{2\times 1 - 1\times 1}{2}$	$\dfrac{2\times 1 - 0\times 1}{2}$
s^2 :	0.5	1
s^1 :	$\dfrac{0.5\times 1 - 1\times 2}{0.5}$	
s^1 :	-3	
s^0 :	$\dfrac{-3\times 1}{-3}$	
s^0 :	1	

Two roots of auxiliary polynomial are lying on the right half of s-plane and the remaining two on the left half of s-plane. The roots of auxiliary equation are also the roots of characteristic polynomial. The two roots lying on the right half of s-plane are indicated by two sign changes in the first column of routh array. The remaining five roots are lying on the left half of s-plane. No roots are lying on imaginary axis.

RESULT

1. The system is unstable.
2. Two roots are lying on right half of s-plane and five roots are lying on left half of s-plane.

METHOD-II

The characteristic equation is, $s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15 = 0$

The given characteristic polynomial is 7th order equation and so it has 7 roots. Since the highest power of s is odd number, form the first row of array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s as shown below.

s^7 :	1	24	24	23 $\quad$ Row-1
s^6 :	9	24	24	15 $\quad$ Row-2

Divide s^6 row by 3 to simplify the computations.

s^7 :	1	24	24	23 $\quad$ Row-1
s^6 :	3	8	8	5 $\quad$ Row-2
s^5 :	1	1	1	$\quad$ Row-3
s^4 :	1	1	1	$\quad$ Row-4
s^3 :	0	0		$\quad$ Row-5

s^5 :	$\dfrac{3\times 24 - 8\times 1}{3}$	$\dfrac{3\times 24 - 8\times 1}{3}$	$\dfrac{3\times 23 - 5\times 1}{3}$
s^5 :	21.33	21.33	21.33
Divide by 21.33			
s^5 :	1	1	1
s^4 :	$\dfrac{1\times 8 - 1\times 3}{1}$	$\dfrac{1\times 8 - 1\times 3}{1}$	$\dfrac{1\times 5 - 0\times 3}{1}$
s^4 :	5	5	5
Divide by 5			
s^4 :	1	1	1
s^3 :	$\dfrac{1\times 1 - 1\times 1}{1}$	$\dfrac{1\times 1 - 1\times 1}{1}$	
s^3 :	0	0	

Since we get a row of zeros, there exists an even polynomial, the even polynomial is nothing but, the auxiliary polynomial.

The auxiliary polynomial is,

$\qquad s^4 + s^2 + 1 = 0$

Divide the characteristic equation by auxiliary polynomial to get the quotient polynomial.

The characteristic polynomial can be expressed as a product of quotient polynomial and auxiliary polynomial.

$$\therefore s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15 = 0$$

$$\Downarrow$$

$$(s^4 + s^2 + 1)(s^3 + 9s^2 + 23s + 15) = 0$$

Even   Quotient polynomial
polynomial

The routh array is constructed for quotient polynomial as shown below:

s^3 :   1   23

s^2 :   9   15

Divide s^2 row by 3,

s^3 :   1   23

s^2 :   3   5

s^1 :   21.33

s^0 :   5

— Column-1

$s^1 : \dfrac{3 \times 23 - 5 \times 1}{3}$

$s^1 : 21.33$

$s^0 : \dfrac{21.33 \times 5 - 0 \times 3}{21.33}$

$s^0 : 5$

The elements of column-1 of quotient polynomial are all positive and there is no sign change. Hence all the roots of quotient polynomial are lying on the left half of s-plane. To determine the stability, the roots of auxiliary polynomial should be evaluated.

The auxiliary equation is, $s^4 + s^2 + 1 = 0$.

Put, $s^2 = x$ in the auxiliary equation. $s^4 + s^2 + 1 = x^2 + x + 1 = 0$

The roots of quadratic are, $x = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\dfrac{1}{2} \pm j\dfrac{\sqrt{3}}{2} = 1\angle 120°$ or $1\angle -120°$

But $s^2 = x$, $\therefore s = \pm\sqrt{x} = \pm\sqrt{1\angle 120°}$ or $\pm\sqrt{1\angle -120°}$

$= \pm\sqrt{1}\angle 120°/2$ or $\pm\sqrt{1}\angle -120°/2$

$= \pm 1\angle 60°$ or $\pm 1\angle -60°$

$= \pm(0.5 + j0.866)$ or $\pm(0.5 - j0.866)$

The roots of auxiliary equation are complex and has quadrantal symmetry. Two roots of auxiliary equation are lying on the right half of s-plane and the other two on the left half of s-plane.

The roots of characteristic equation are given by the roots of auxiliary polynomial and the roots of quotient polynomial. Hence we can conclude that two roots of characteristic equation are lying on the right half of s-plane and so system is unstable. The remaining five roots are lying on left half of s-plane.

EXAMPLE 4.6

The characteristic polynomial of a system is $s^7 + 5s^6 + 9s^5 + 9s^4 + 4s^3 + 20s^2 + 36s + 36 = 0$. Determine the location of roots on the s-plane and hence the stability of the system.

SOLUTION

The characteristic equation is, $s^7 + 5s^6 + 9s^5 + 9s^4 + 4s^3 + 20s^2 + 36s + 36 = 0$.

The given characteristic polynomial is 7th order equation and so it has 7 roots. Since the highest power of s is odd number, form the first row of array using the coefficients of odd powers of s and form the second row of array using the coefficients of even powers of s as shown below.

$$s^7 \; : \quad 1 \quad 9 \quad 4 \quad 36 \qquad \text{.... Row-1}$$
$$s^6 \; : \quad 5 \quad 9 \quad 20 \quad 36 \qquad \text{.... Row-2}$$

Divide s^6 row by 5 to simplify the computations.

$$s^7 \; : \quad 1 \quad 9 \quad 4 \quad 36 \qquad \text{.... Row-1}$$
$$s^6 \; : \quad 1 \quad 1.8 \quad 4 \quad 7.2 \qquad \text{.... Row-2}$$
$$s^5 \; : \quad 1 \quad 0 \quad 4 \qquad \text{.... Row-3}$$
$$s^4 \; : \quad 1 \quad 0 \quad 4 \qquad \text{.... Row-4}$$
$$s^3 \; : \quad 0 \quad 0 \qquad \text{.... Row-5}$$

The row of all zeros indicate the existence of even polynomial, which is also the auxiliary polynomial. The auxiliary polynomial is, $s^4 + 4 = 0$. Divide the characteristic equation by auxiliary equation to get the quotient polynomial.

The characteristic equation can be expressed as a product of quotient polynomial and auxiliary equation.

$$\therefore s^7 + 5s^6 + 9s^5 + 9s^4 + 4s^3 + 20s^2 + 36s + 36 = 0$$

$$(s^4 + 4) \qquad (s^3 + 5s^2 + 9s + 9) = 0$$

Even polynomial   Quotient polynomial

The routh array is constructed for quotient polynomial as shown below.

$$s^3 \; : \quad 1 \quad | \quad 9$$
$$s^2 \; : \quad 5 \quad | \quad 9$$
$$s^1 \; : \quad 7.2$$
$$s^0 \; : \quad 9$$

Column-1

$$s^1 : \frac{5 \times 9 - 9 \times 1}{5}$$
$$s^1 : \quad 7.2$$
$$s^0 : \frac{7.2 \times 9 - 0 \times 5}{7.2}$$
$$s^0 : \quad 9$$

$$s^5 : \frac{1 \times 9 - 18 \times 1}{1} \quad \frac{1 \times 4 - 4 \times 1}{1} \quad \frac{1 \times 36 - 7.2 \times 1}{1}$$
$$s^5 : 7.2 \qquad\qquad 0 \qquad\qquad 28.8$$

Divide by 7.2

$$s^5 : \quad 1 \qquad\qquad 0 \qquad\qquad 4$$

$$s^4 : \frac{1 \times 18 - 0 \times 1}{1} \quad \frac{1 \times 4 - 4 \times 1}{1} \quad \frac{1 \times 7.2 - 0 \times 1}{1}$$
$$s^4 : 1.8 \qquad\qquad 0 \qquad\qquad 7.2$$

Divide by 1.8

$$s^4 : \quad 1 \qquad\qquad 0 \qquad\qquad 4$$

$$s^3 : \frac{1 \times 0 - 0 \times 1}{1} \quad \frac{1 \times 4 - 4 \times 1}{1}$$
$$s^3 : \quad 0 \qquad\qquad 0$$

There is no sign change in the elements of first column of routh array of quotient polynomial. Hence all the roots of quotient polynomial are lying on the left half of s-plane.

To determine the stability, the roots of auxiliary polynomial should be evaluated.

The auxiliary polynomial is, $s^4 + 4 = 0$.

Put, $s^2 = x$ in the auxiliary equation, $\therefore s^4 + 4 = x^2 + 4 = 0$

$$\therefore x^2 = -4 \quad \Rightarrow \quad x = \pm\sqrt{-4} = \pm j2 = 2\angle 90° \text{ or } 2\angle -90°$$

But, $s = \pm\sqrt{x} = \pm\sqrt{2\angle 90°}$ or $\pm\sqrt{2\angle -90°} = \pm\sqrt{2}\angle 90°/2$ or $\pm\sqrt{2}\angle -90°/2$

$$= \pm\sqrt{2}\angle 45° \quad \text{or} \quad \pm\sqrt{2}\angle -45 = \pm(1+j1) \text{ or } \pm(1-j1)$$

The roots of auxiliary equation are complex and has quadrantal symmetry. Two roots of auxiliary equation are lying on the right half of s-plane and the other two on the left half of s-plane.

The roots of characteristic equation are given by roots of quotient polynomial and auxiliary polynomial. Hence we can conclude that two roots of characteristic equation are lying on the right half of s-plane and so the system is unstable. The remaining five roots are lying on the left half of s-plane.

RESULT

(a) The system is unstable.

(b) Two roots are lying on the right half of s-plane and five roots are lying on the left half of s-plane.

EXAMPLE 4.7

Use the routh stability criterion to determine the location of roots on the s-plane and hence the stability for the system represented by the characteristic equation $s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$.

SOLUTION

The characteristic equation of the system is, $s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$.

The given characteristic polynomial is 5th order equation and so it has 5 roots. Since the highest power of s is odd number, form the first row of routh array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

$$s^5 \quad : \quad 1 \quad 8 \quad 7 \qquad ....\text{Row-1}$$
$$s^4 \quad : \quad 4 \quad 8 \quad 4 \qquad ....\text{Row-2}$$

Divide s^4 row by 4 to simplify the calculations.

$$s^5 \quad : \quad 1 \quad 8 \quad 7 \qquad ....\text{Row-1}$$
$$s^4 \quad : \quad 1 \quad 2 \quad 1 \qquad ....\text{Row-2}$$
$$s^3 \quad : \quad 1 \quad 1 \qquad ....\text{Row-3}$$
$$s^2 \quad : \quad 1 \quad 1 \qquad ....\text{Row-4}$$
$$s^1 \quad : \quad \in \qquad ....\text{Row-5}$$
$$s^0 \quad : \quad 1 \qquad ....\text{Row-6}$$

Column-1

When $\in \rightarrow 0$, there is no sign change in the first column of routh array. But we have a row of all zeros (s^1 row or row-5) and so there is a possibility of roots on imaginary axis. This can be found from the roots of auxiliary polynomial. Here the auxiliary polynomial is given by s^2 row.

$$s^3 : \frac{1 \times 8 - 2 \times 1}{1} \quad \frac{1 \times 7 - 1 \times 1}{1}$$
$$s^3 : 6 \qquad\qquad 6$$
Divide by 6
$$s^3 : 1 \qquad\qquad 1$$

$$s^2 : \frac{1 \times 2 - 1 \times 1}{1} \quad \frac{1 \times 1 - 0 \times 1}{1}$$
$$s^2 : 1 \qquad\qquad 1$$

$$s^1 : \frac{1 \times 1 - 1 \times 1}{1}$$
$$s^1 : 0$$
Let $0 \rightarrow \in$
$$s^1 : \in$$

$$s^0 : \frac{\in \times 1 - 0 \times 1}{\in}$$
$$s^0 : 1$$

The auxiliary polynomial is, $s^2 + 1 = 0$; $\quad \therefore s^2 = -1$ or $s = \pm\sqrt{-1} = \pm j1$

The roots of auxiliary polynomial are $+j1$ and $-j1$, lying on imaginary axis. The roots of auxiliary polynomial are also roots of characteristic equation. Hence two roots of characteristic equation are lying on imaginary axis and so the system is limitedly or marginally stable. The remaining three roots of characteristic equation are lying on the left half of s-plane.

RESULT

(a) The system is limitedly or marginally stable.

(b) Two roots are lying on imaginary axis and three roots are lying on left half of s-plane.

EXAMPLE 4.8

Use the routh stability criterion to determine the location of roots on the s-plane and hence the stability for the system represented by the characteristic equation, $s^6 + s^5 + 3s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$.

SOLUTION

The characteristic polynomial of the system is, $s^6 + s^5 + 3s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$.

The given characteristic polynomial is 6th order equation and so it has 6 roots. Since the highest power of s is even number, form the first row of routh array using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s as shown below.

s^6	:	1	3　3　1		.... Row-1
s^5	:	1	3　2		.... Row-2
s^4	:	$\in$	1　1		.... Row-3
s^3	:	$\dfrac{3\in-1}{\in}$	$\dfrac{2\in-1}{\in}$		.... Row-4
s^2	:	$\dfrac{-2\in^2+4\in-1}{3\in-1}$ 1			.... Row-5
s^1	:	$\dfrac{4\in^2-\in}{2\in^2-4\in+1}$			.... Row-6
s^0	:	1			.... Row-7

$$s^4 : \frac{1\times3-3\times1}{1} \quad \frac{1\times3-2\times1}{1} \quad \frac{1\times1-0\times1}{1}$$

$$s^4 : 0 \qquad 1 \qquad 1$$

let $0 \to \in$

$$s^4 : \quad \in \qquad 1 \qquad 1$$

$$s^3 : \frac{\in\times3-1\times1}{\in} \quad \frac{\in\times2-1\times1}{\in}$$

$$s^3 : \frac{3\in-1}{\in} \quad \frac{2\in-1}{\in}$$

$$s^2 : \frac{\dfrac{3\in-1}{\in}-\dfrac{2\in-1}{\in}\times\in}{\dfrac{3\in-1}{\in}} \quad \frac{\dfrac{3\in-1}{\in}\times1-0\times\in}{\dfrac{3\in-1}{\in}}$$

$$s^2 : \frac{-2\in^2+4\in-1}{3\in-1} \qquad 1$$

On letting $\in \to 0$, we get,

s^6	:	1	3　3　1	.... Row-1
s^5	:	1	3　2	.... Row-2
s^4	:	0	1　1	.... Row-3
s^3	:	$-\infty$	$-\infty$	.... Row-4
s^2	:	1	1	.... Row-5
s^1	:	0		.... Row-6
s^0	:	1		.... Row-7

$$s^1 : \frac{\dfrac{-2\in^2+4\in-1}{3\in-1}\times\dfrac{2\in-1}{\in}-\dfrac{3\in-1}{\in}\times1}{\dfrac{-2\in^2+4\in-1}{3\in-1}}$$

$$s^1 : \frac{(-2\in^2+4\in-1)(2\in-1)-(3\in-1)(3\in-1)}{\in(-2\in^2+4\in-1)}$$

$$s^1 : \frac{-4\in^3+\in^2}{\in(-2\in^2+4\in-1)} = \frac{4\in^2-\in}{2\in^2-4\in+1}$$

$$s^0 : \frac{\dfrac{4\in^2-\in}{4\in^2-4\in+1}\times1-0\times\dfrac{-2\in^2+4\in-1}{3\in-1}}{(4\in^2-\in)/(4\in^2-4\in+1)}$$

$$s^0 : 1$$

Since there is a row of all zeros (s^1 row) there is a possibility of roots on imaginary axis. The auxiliary polynomial is $s^2 + 1 = 0$.

The roots of auxiliary polynomial are, $s = \pm\sqrt{-1} = \pm j1$

The roots of auxiliary polynomial are also roots of characteristic equation. Hence two roots are lying on imaginary axis. Therefore divide the characteristic polynomial by auxiliary equation and construct the routh array for quotient polynomial to find the roots lying on right half of s-plane.

The characteristic polynomial can be expressed as a product of auxiliary polynomial and quotient polynomial.

$$\therefore s^6+s^5+3s^4+3s^3+3s^2+2s+1 = 0 \Rightarrow (s^2+1)(s^4+s^3+2s^2+2s+1)=0$$

Even polynomial　　　Quotient polynomial

The routh array for quotient polynomial is constructed as shown below.

s^4	:	1	2　1		.... Row-1
s^3	:	1	2		.... Row-2
s^2	:	$\in$	1		.... Row-3
s^1	:	$\dfrac{2\in-1}{\in}$			.... Row-4
s^0	:	1			.... Row-5

$$s^2 : \frac{1\times2-2\times1}{1} \quad \frac{1\times1-0\times1}{1}$$

$$s^2 : 0 \qquad 1$$

let $0 \to \in$

$$s^2 : \in \quad 1$$

$$s^1 : \frac{\in\times2-1\times1}{\in}$$

$$s^1 : \frac{2\in-1}{\in}$$

$$s^0 : \frac{\dfrac{2\in-1}{\in}\times1-0\times\in}{(2\in-1)/\in}$$

$$s^0 : 1$$

On letting $\in \to 0$, we get

s^4	:	1	2	1	.... Row-1
s^3	:	1	2		.... Row-2
s^2	:	0	1		.... Row-3
s^1	:	$-\infty$			.... Row-4
s^0	:	1			.... Row-5

Column-1

The division panel:

$$s^2 + 1 \overline{)\, s^6 + s^5 + 3s^4 + 3s^3 + 3s^2 + 2s + 1} \quad \text{(Quotient polynomial } s^4 + s^3 + 2s^2 + 2s + 1)$$

$$\underline{s^6 \; (-) \; + \; (-) \, s^4}$$
$$s^5 + 2s^4 + 3s^3 + 3s^2 + 2s + 1$$
$$\underline{(-)s^5 \qquad (-)+s^3}$$
$$2s^4 + 2s^3 + 3s^2 + 2s + 1$$
$$\underline{(-)2s^4 \qquad (-)+2s^2}$$
$$2s^3 + s^2 + 2s + 1$$
$$\underline{(-)2s^3 \quad (-)+2s}$$
$$s^2 \qquad +1$$
$$\underline{(-)s^2 \quad (-)+1}$$
$$0$$

(Even polynomial)

On examining the first column of the routh array of quotient polynomial, we found that there are two sign changes. Hence two roots are lying on the right half of s-plane and other two roots of quotient polynomial are lying on the left half of s-plane.

The roots of characteristic equation are given by roots of auxiliary polynomial and quotient polynomial. Hence two roots are lying on imaginary axis, two roots are lying on right half of s-plane and the remaining two roots are lying on left half of s-plane. Hence the system is unstable.

RESULT

(a) The system is unstable.

(b) Two roots are lying on imaginary axis, two roots are lying on right half of s-plane and two roots are lying on left half of s-plane.

EXAMPLE 4.9

Determine the range of K for stability of unity feedback system whose open loop transfer function is

$$G(s) = \frac{K}{s(s + 1)(s + 2)} \; .$$

SOLUTION

The closed loop transfer function, $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1 + G(s)} = \dfrac{\dfrac{K}{s(s+1)(s+2)}}{1 + \dfrac{K}{s(s+1)(s+2)}} = \dfrac{K}{s(s+1)(s+2) + K}$

The characteristic equation is, $\quad s(s+1)(s+2) + K = 0$

$$\therefore s(s^2 + 3s + 2) + K = 0 \quad \Rightarrow \quad s^3 + 3s^2 + 2s + K = 0$$

The routh array is constructed as shown below.

The highest power of s in the characteristic polynomial is odd number. Hence form the first row using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

s^3	:	1	2
s^2	:	3	K
s^1	:	$\dfrac{6-K}{3}$	
s^0	:	K	

Column-1

$$s^1 : \frac{3 \times 2 - K \times 1}{3}$$
$$s^1 : \frac{6 - K}{3}$$

$$s^0 : \frac{\dfrac{6-K}{3} \times K - 0 \times 3}{(6-K)/3}$$
$$s^0 : K$$

For the system to be stable there should not be any sign change in the elements of first column. Hence choose the value of K so that the first column elements are positive.

From s^0 row, for the system to be stable, $K > 0$

From s^1 row, for the system to be stable, $\dfrac{6-K}{3} > 0$

For $\dfrac{6-K}{3} > 0$, the value of K should be less than 6.

∴ The range of K for the system to be stable is $0 < K < 6$.

RESULT

The value of K is in the range $0 < K < 6$ for the system to be stable.

EXAMPLE 4.10

The open loop transfer function of a unity feedback control system is given by,

$$G(s) = \frac{K}{(s+2)\,(s+4)\,(s^2+6s+25)}.$$

By applying the routh criterion, discuss the stability of the closed-loop system as a function of K. Determine the value of K which will cause sustained oscillations in the closed-loop system. What are the corresponding oscillating frequencies?

SOLUTION

The closed loop transfer function } $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)} = \dfrac{\dfrac{K}{(s+2)\,(s+4)\,(s^2+6s+25)}}{1+\dfrac{K}{(s+2)\,(s+4)\,(s^2+6s+25)}} = \dfrac{K}{(s+2)\,(s+4)\,(s^2+6s+25)+K}$

The characteristic equation is given by the denominator polynomial of closed loop transfer function.

The characteristic equation is, $(s+2)\,(s+4)\,(s^2+6s+25)+K = 0$.

∴ $(s^2+6s+8)\,(s^2+6s+25)+K = 0$ ⇒ $s^4+12s^3+69s^2+198s+200+K = 0$

The routh array is constructed as shown below. The highest power of s in the characteristic equation is even number. Hence form the first row using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

s^4 :	1	69	200+K Row-1
s^3 :	12	198	..... Row-2

$s^2 : \dfrac{1 \times 69 - 16.5 \times 1}{1} \quad \dfrac{1 \times (200+K)}{1}$

$s^2 : 52.5 \qquad\qquad 200 + K$

Divide s^3 row by 12 to simplify the calculations

s^4 :	1	69	200+K	..... Row-1
s^3 :	1	16.5		..... Row-2
s^2 :	52.5	200+K		..... Row-3
s^1 :	$\dfrac{666.25-K}{52.5}$			..... Row-4
s^0 :	200+K			..... Row-5

└─ Column-1

$s^1 : \dfrac{52.5 \times 16.5 - (200+K) \times 1}{52.5}$

$s^1 : \dfrac{666.25-K}{52.5}$

$s^0 : \dfrac{\dfrac{666.25-K}{52.5} \times (200+K)}{(666.25-K)/52.5}$

$s^0 : 200 + K$

For the system to be stable there should not be any sign change in the elements of first column. Hence choose the value of K so that the first column elements are positive.

From s^1 row, for the system to be stable, (666.25-K) > 0.

Since (666.25-K) > 0, should be less than 666.25.

From s^0 row, for the system to be stable, (200+K) > 0

Since (200+K) > 0, K should be greater than -200, but practical values of K starts from 0. Hence K should be greater than zero.

∴ The range of K for the system to be stable is 0<K<666.25.

When K= 666.25 the s^1 row becomes zero, which indicates the possibility of roots on imaginary axis. A system will oscillate if it has roots on imaginary axis and no roots on right half of s-plane.

When K=666.25, the coefficients of auxiliary equation are given by the s^2 row.

∴ The auxiliary equation is, $52.5s^2+200+K=0$

$$52.5s^2 + 200 + 666.25 = 0$$
$$s^2 = \frac{-200 - 666.25}{52.5} = -16.5$$
$$s = \pm\sqrt{-16.5} = \pm j\sqrt{16.5} = \pm j4.06$$

When K = 666.25, the system has roots on imaginary axis and so it oscillates. The frequency of oscillation is given by the value of root on imaginary axis.

∴ The frequency of oscillation, $\omega = 4.06$ rad/sec.

RESULT

(a) The range of K for stability is 0<K<666.25
(b) The system oscillates when K = 666.25
(c) The frequency of oscillation, $\omega = 4.06$ rad/sec. (When K = 666.25).

EXAMPLE 4.11

The open loop transfer function of a unity feedback system is given by, $G(s) = \dfrac{K(s+1)}{s^3 + as^2 + 2s + 1}$. Determine the value of K and a so that the system oscillates at a frequency of 2 rad/sec.

SOLUTION

The closed loop transfer function $\begin{cases} \end{cases}$ $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)} = \dfrac{\dfrac{K(s+1)}{s^3 + as^2 + 2s + 1}}{1+\dfrac{K(s+1)}{s^3 + as^2 + 2s + 1}} = \dfrac{K(s+1)}{s^3 + as^2 + 2s + 1 + K(s+1)}$

The characteristic equation is, $s^3 + as^2 + 2s + 1 + K(s+1) = 0$.

$$s^3 + as^2 + 2s + 1 + Ks + K = 0 \quad \Rightarrow \quad s^3 + as^2 + (2+K)s + 1 + K = 0$$

The routh array of characteristic polynomial is constructed as shown below. The maximum power of s is odd, hence the first row of routh array is formed using coefficients of odd powers of s and the second row of routh array is formed using coefficients of even powers of s.

If the elements of s^1 row are all zeros then there exists an even polynomial (or auxiliary polynomial). If the roots of the auxiliary polynomial are purely imaginary then the roots are lying on imaginary axis and the system oscillates. The frequency of oscillation is the root of auxiliary polynomial.

Routh array

s^3 :   1   2+K

s^2 :   a   1+K

s^1 : $\dfrac{a(2+K)-(1+K)}{a}$

s^0 :   1+K

From s^2 row, the auxiliary polynomial is,

$$as^2 + (1+K) = 0 \quad \Rightarrow \quad as^2 = -(1+K) \quad \Rightarrow \quad s = \pm j\sqrt{\frac{1+K}{a}}$$

Given that, $s = \pm j2$, $\therefore \sqrt{\dfrac{1+K}{a}} = 2 \quad \Rightarrow \quad \dfrac{1+K}{a} = 4 \quad \Rightarrow \quad K = 4a - 1$

From s^1 row, $\dfrac{a(2+K)-(1+K)}{a} = 0 \Rightarrow a(2+K) - (1+K) = 0 \Rightarrow 2a + Ka - 1 - K = 0$

$\therefore \quad 2a - 1 + K(a-1) = 0$

Put, K = 4a − 1

$\therefore 2a - 1 + (4a-1)(a-1) = 0 \Rightarrow 2a - 1 + 4a^2 - 4a - a + 1 = 0 \Rightarrow 4a^2 - 3a = 0$ (or) a(4a−3)=0

Since $a \neq 0$,   4a − 3 = 0, $\therefore a = 3/4$

When a = (3 / 4),   K = 4a −1 = 4 × (3 / 4) − 1 = 2

RESULT

When the system oscillates at a frequency of 2 rad/sec, K = 2   and a = 3/4.

EXAMPLE 4.12

A feedback system has open loop transfer function of $G(s) = \dfrac{Ke^{-s}}{s(s^2 + 5s + 9)}$. Determine the maximum value of K for stability of closed loop system.

SOLUTION

Generally control systems have very low bandwidth which implies that it has very low frequency range of operation. Hence for low frequency ranges the term e^{-s} can be replaced by, 1− s, (i.e., $e^{s-T} \approx 1 - sT$).

$$\therefore G(s) = \frac{K e^{-s}}{s(s^2 + 5s + 9)} \approx \frac{K(1-s)}{s(s^2 + 5s + 9)}$$

The closed loop transfer function $\left\{ \dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)} = \dfrac{\dfrac{K(1-s)}{s(s^2+5s+9)}}{1 + \dfrac{K(1-s)}{s(s^2+5s+9)}} = \dfrac{K(1-s)}{s(s^2+5s+9)+K(1-s)} \right.$

The characteristic equation is given by the denominator polynomial of closed loop transfer function.

$\therefore$ The characteristic equation is, $s(s^2 + 5s + 9) + K(1-s) = 0$

$\therefore s(s^2 + 5s + 9) + K(1-s) = s^3 + 5s^2 + 9s + K - Ks = 0 \Rightarrow s^3 + 5s^2 + (9-K)s + K = 0$

The routh array of characteristic polynomial is constructed as shown below.

The maximum power of s in the characteristic polynomial is odd, hence form the first row of routh array using coefficients of odd powers of s and second row of routh array using coefficients of even powers of s.

s^3 : 1   9 – K

s^2 : 5   K

s^1 : 9 – 1.2K

s^0 : K

s^1 :	$\dfrac{5 \times (9-K) - K \times 1}{5}$
s^1 :	$\dfrac{45 - 5K - K}{5}$
s^1 :	$\dfrac{45 - 6K}{5} \approx 9 - 12K$
s^0 :	$\dfrac{(9 - 12K) \times K}{(9 - 12K)}$
s^0 : K	

From s^1 row, for stability of the system, (9 – 1.2K) > 0

If (9 – 1.2K) > 0 then 1.2K < 9 ; $\therefore K < \dfrac{9}{12} = 7.5$

From s^0 row, for stability of the system, K > 0

Finally we can conclude that for stability of the system K should be in the range of 0 < K < 7.5

RESULT

For stability of the system K should be in the range of, 0 < K < 7.5 .

4.4 MATHEMATICAL PRELIMINARIES FOR NYQUIST STABILITY CRITERION

Let F(s) be a function of s, which is expressed as a ratio of two polynomials in s, as shown in equation (4.14), (the polynomials are expressed in the factored form).

$$F(s) = \frac{(s - z_1)(s - z_2) \dots\dots (s - z_m)}{(s - p_1)(s - p_2) \dots\dots (s - p_n)} \qquad \dots(4.14)$$

The roots of numerator polynomial are zeros and the roots of denominator polynomial are poles. The function has m number of zeros and n number of poles.

Here, s is a complex variable expressed as, $s = \sigma + j\omega$, where s is real part of σ and ω is imaginary part of s. (The s is also called complex frequency). For a particular value of σ and ω, the s will represent a point in the s-plane.

Since s is a complex variable, the function F(s) will also be a complex quantity for any value of s. Hence, F(s) can also be expressed as, F(s) = u+jv, where u is real part of F(s) and v is imaginary part of F(s). Let us define another complex plane called F(s)-plane, with coordinates u and v. For a particular value of s, the F(s) will represent a point in F(s)-plane.

Therefore, for every point s in the s-plane at which F(s) is analytic, there exists a corresponding point F(s) in the F(s)-plane. Hence it can be concluded that the function F(s) maps the points in the s-plane into the F(s)-plane.

> Note : A function is analytic in the s-plane provided the function and all its derivatives exist. The points in the s-plane where the function (or it derivatives) does not exist are called singular points.

Since any number of points of analycity in the s-plane can be mapped into the F(s)- plane it can be concluded that for a contour in the s-plane which does not go through any singular point, there exists a corresponding contour in the F(s)-plane as shown in fig 4.2.

The table 4.2 shows examples of arbitrary s-plane contours and their corresponding F(s)-plane contours (exact shape is not shown).