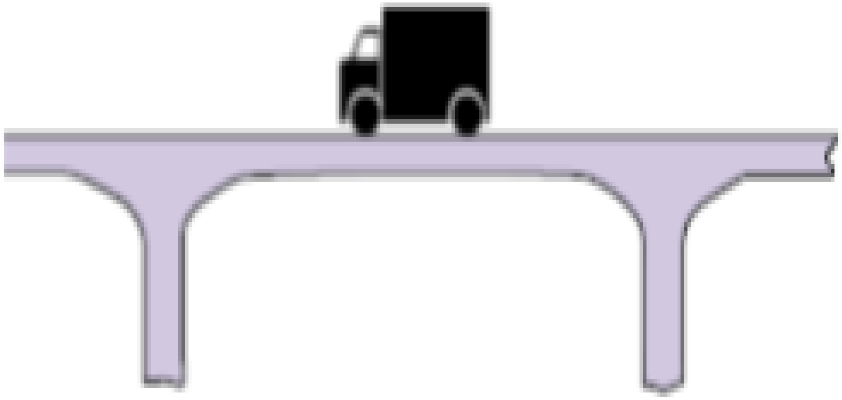
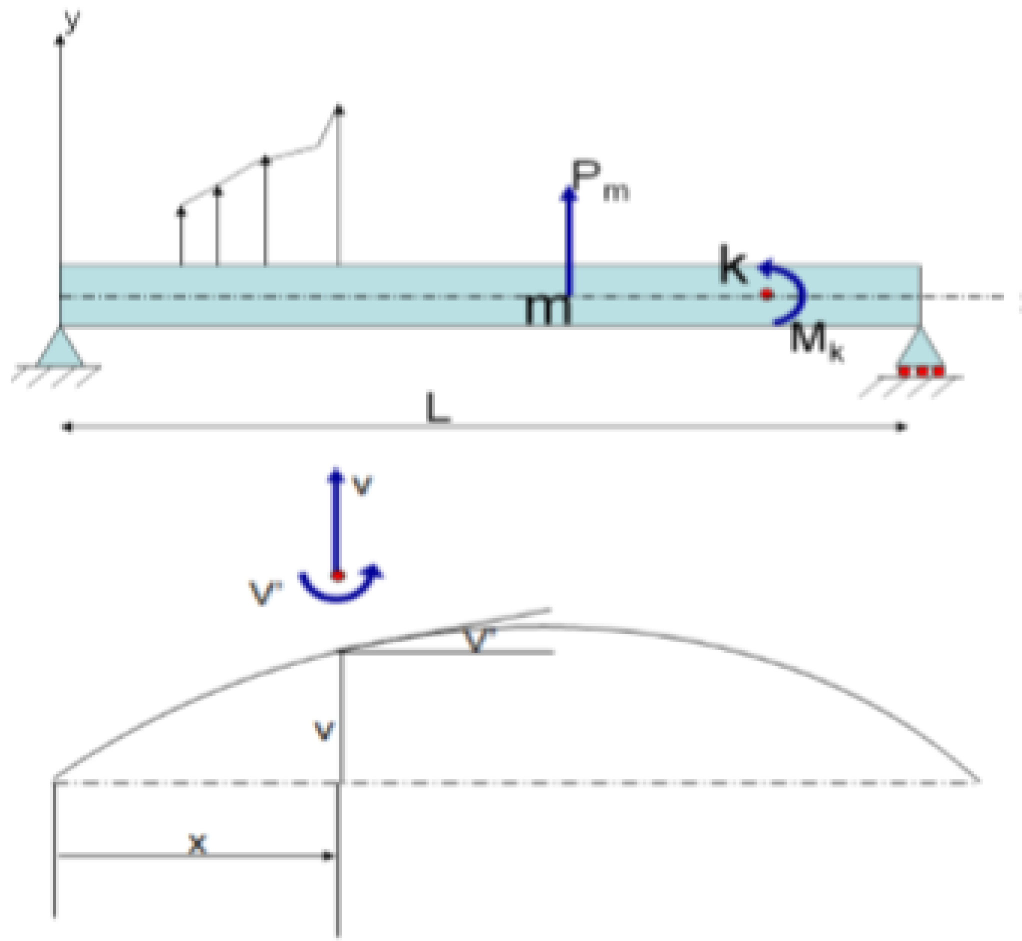
Analysis of Beam element-Module 3

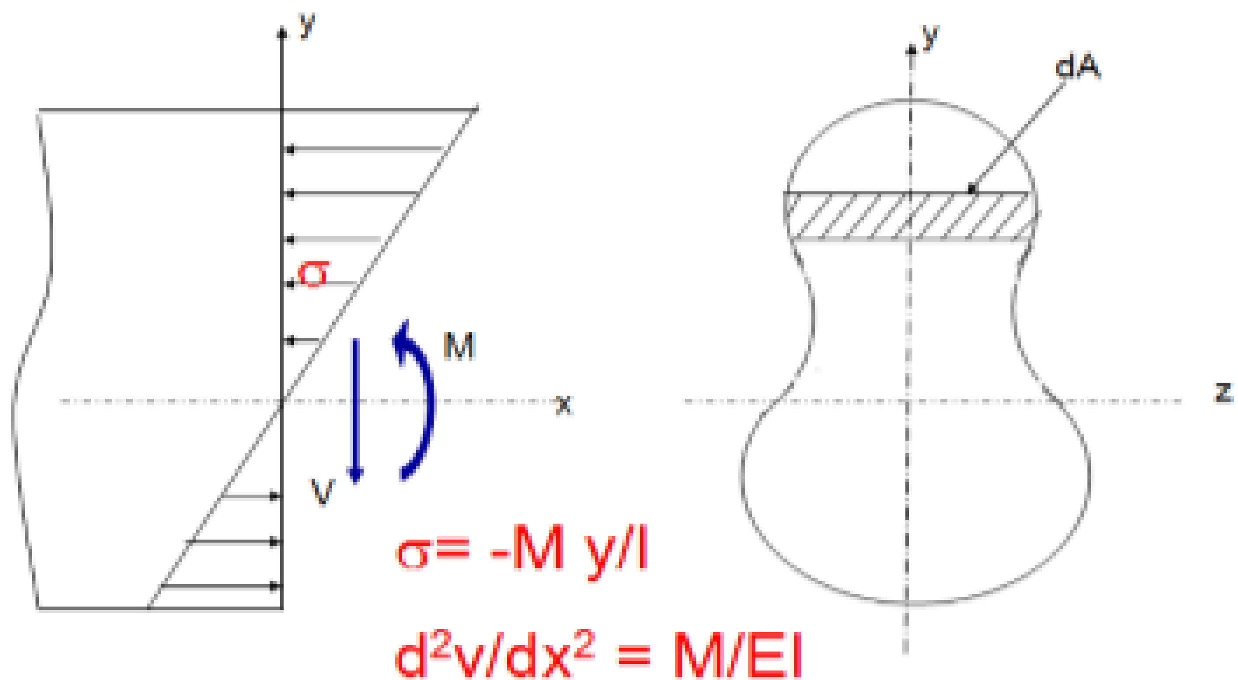
Beam is a structural member which is acted upon by a system of external loads perpendicular to axis which causes bending that is deformation of bar produced by perpendicular load as well as force couples acting in a plane. Beams are the most common type of structural component, particularly in Civil and Mechanical Engineering. A beam is a bar-like structural member whose primary function is to support transverse loading and carry it to the supports



A truss and a bar undergoes only axial deformation and it is assumed that the entire cross section undergoes the same displacement, but beam on other hand undergoes transverse deflection denoted by v. Fig shows a beam subjected to system of forces and the deformation of the neutral axis



We assume that cross section is doubly symmetric and bending take place in a plane of symmetry. From the strength of materials we observe the distribution of stress as shown.



Where M is bending moment and I is the moment of inertia. According to the Euler Bernoulli theory. The entire c/s has the same transverse deflection V as the neutral axis, sections originally perpendicular to neutral axis remain plane even after bending

Potential energy approach :

Strain energy in an element for a length dx is given by

$$= \frac{1}{2} \int_A \sigma\, \varepsilon \, dA \, dx$$

$$= \frac{1}{2} \int_A \sigma\, \sigma/E \, dA \, dx$$

$$= \frac{1}{2} \int_A \sigma^2/E \, dA \, dx$$

But we know $\sigma = M\, y / I$ substituting this in above equation we get.

$$= \frac{1}{2} \int_A \frac{M^2}{EI^2} y^2 dA \, dx$$

$$= \frac{1}{2} \frac{M^2}{EI^2} \left[\int_A y^2 dA \right] dx$$

$$= \frac{1}{2} \frac{M^2}{EI} dx$$

But

Therefore strain energy for an element is given by

$$M = EI\, d^2v/dx^2$$

$$= \tfrac{1}{2}\int_0^L EI\,(d^2v/dx^2)^2\, dx$$

Now the potential energy for a beam element can be written as

$$\Pi = \tfrac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx - \int_0^L p\, v\, dx - \sum_m P_m V_m - \sum_k M_k V'_k$$

P ---- distribution load per unit length

P_m----- point load @ point m

V_m----- deflection @ point m

M_k----- momentum of couple applied at point k

V'_k----- slope @ point k

Hermite shape functions:

1D linear beam element has two end nodes and at each node 2 dof which are denoted as Q2i-1 and Q2i at node i. Here Q2i-1 represents transverse deflection where as Q2i is slope or rotation. Consider a beam element has node 1 and 2 having dof as shown.

The shape functions of beam element are called as **Hermite shape functions** as they contain both nodal value and nodal slope which is satisfied by taking polynomial of cubic order

$$H_i = a_i + b_i\,\xi + c_i\,\xi^2 + d_i\,\xi^3$$

that must satisfy the following conditions

ξ	H_1	H_1'	H_2	H_2'	H_3	H_3'	H_4	H_4'
$\xi = -1$	1	0	0	1	0	0	0	0
$\xi = 1$	0	0	0	0	1	0	0	1

Applying these conditions determine values of constants as

$$H_i = a_i + b_i\,\xi + c_i\,\xi^2 + d_i\,\xi^3$$

@ node 1

$$H_1 = 1,\quad H_1' = 0\,,\quad \xi = -1$$

$$1 = a_1 - b_1 + c_1 - d_1 \longrightarrow (1)$$

$$H_1' = \frac{dH_1}{d\xi} = 0 = b_1 - 2c_1 + 3d_1 \longrightarrow (2)$$

$$H_i = a_i + b_i\,\xi + c_i\,\xi^2 + d_i\,\xi^3$$

@ node 2

$$H_1 = 0,\quad H_1' = 0\,,\quad \xi = 1$$

$$0 = a_1 + b_1 + c_1 + d_1 \longrightarrow (3)$$

$$H_1' = \frac{dH_1}{d\xi} = 0 = b_1 + 2c_1 + 3d_1 \longrightarrow (4)$$

Solving above 4 equations we have the values of constants

$$1 = a_1 - b_1 + c_1 - d_1 \longrightarrow (1)$$

$$0 = b_1 - 2c_1 + 3d_1 \longrightarrow (2)$$

$$0 = a_1 + b_1 + c_1 + d_1 \longrightarrow (3)$$

$$0 = b_1 + 2c_1 + 3d_1 \longrightarrow (4)$$

Adding equation 1 & equation 3

1 = 2 (a1 + c1)

1/2 = (a1 + c1)......(5)

Substitute eq.1 & eq.2

1= a1 + c1 – b1 – d1

1 = ½ - b1 –d1

b1 + d1 = -1/2

-3d1 + d1 = -1/2

d1 = 1/4

subtracting equation 2 & equation 4

0 = -4c1

c1 = 0

0 = b1 + 2c1 + 3d1

b1 = - 3d1 ....(6)

Substitute b1 value in eq, we get

Substitute c1 value in eq,.5 we get

1/2 = (a1 + c1)......(5)

a1 = 1/2

Substitute d1 value in eq,.6 we get

b1 = -3/4

Therefore

$a_1 = \frac{1}{2}$, $b_1 = -\frac{3}{4}$, $c_1 = 0$, $d_1 = \frac{1}{4}$

$$H_1 = a_1 + b_1 \xi + c_1 \xi^2 + d_1 \xi^3$$

$$H_1 = a_1 + b_1 \xi + c_1 \xi^2 + d_1 \xi^3$$

$$= \frac{1}{2} + \left(-\frac{3}{4}\right)\xi + (0)\xi^2 + \left(\frac{1}{4}\right)\xi^3$$

$$= \frac{1}{2} - \frac{3}{4}\xi + \frac{1}{4}\xi^3$$

$$H_1 = \frac{1}{4}\left(2 - 3\xi + \xi^3\right)$$

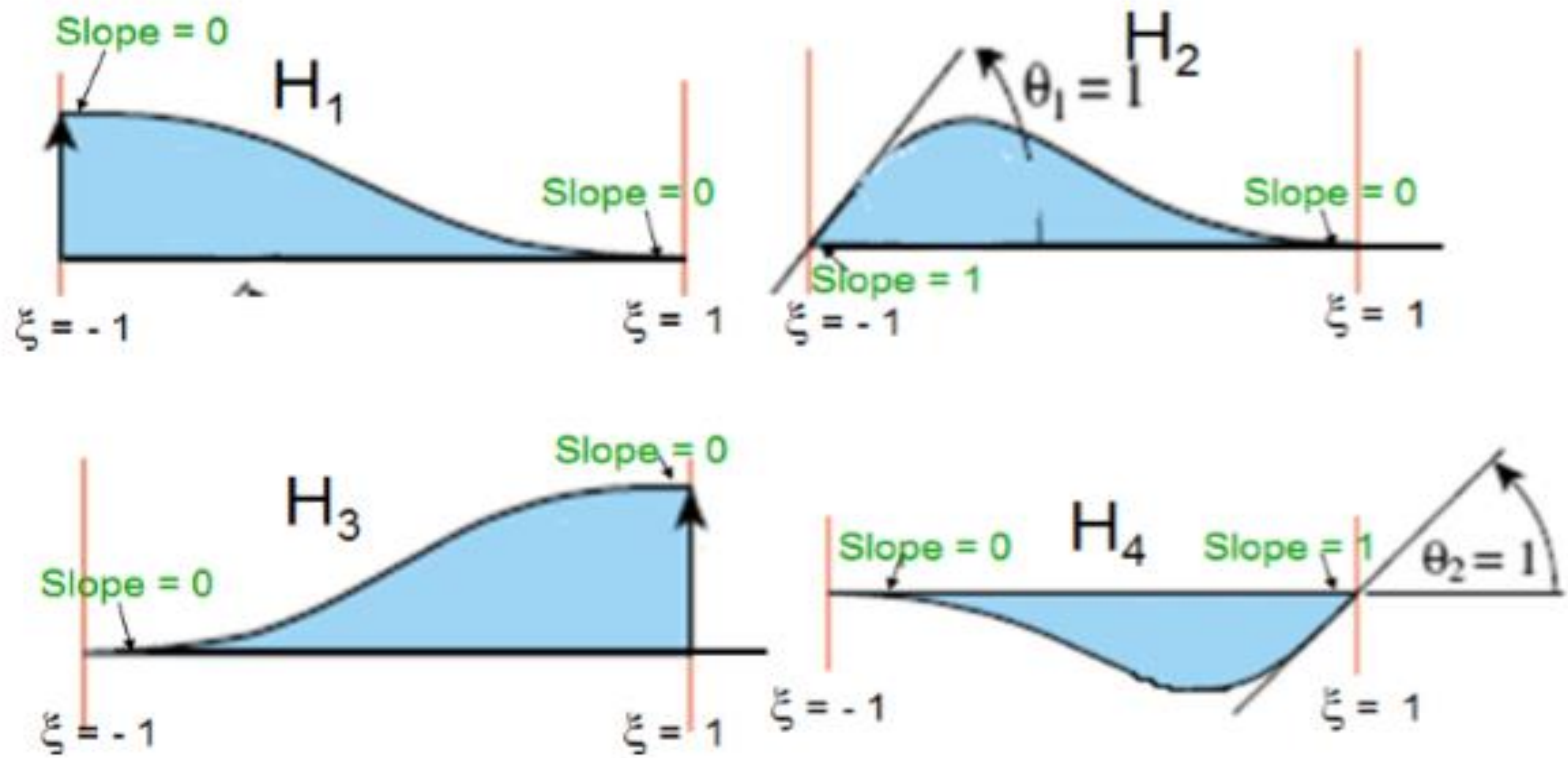
Similarly we can derive

$$H_2 = \tfrac{1}{4}(1 - \xi - \xi^2 + \xi^3) \qquad H_1 = \tfrac{1}{4}(2 - 3\xi + \xi^3)$$

$$H_3 = \tfrac{1}{4}(2 + 3\xi - \xi^3)$$

$$H_4 = \tfrac{1}{4}(-1 - \xi + \xi^2 + \xi^3)$$

Following graph shows the variations of Hermite shape functions



Element Stiffness matrix for Beam :

Once the shape functions are derived

$$H_1 = \tfrac{1}{4}(2 - 3\xi + \xi^3)$$

$$H_2 = \tfrac{1}{4}(1 - \xi - \xi^2 + \xi^3)$$

$$H_3 = \tfrac{1}{4}(2 + 3\xi - \xi^3)$$

$$H_4 = \tfrac{1}{4}(-1 - \xi + \xi^2 + \xi^3)$$

we can write the equation of the form

$$V(\xi) = H_1V_1 + H_2\left(\frac{dv}{d\xi}\right)_1 + H_3V_3 + H_4\left(\frac{dv}{d\xi}\right)_2$$

But

$$\frac{dv}{d\xi} = \frac{dv}{dx}\,\frac{dx}{d\xi} = \frac{dv}{dx}\,\frac{Le}{2}$$

ie

$$V(\xi) = H_1V_1 + H_2\left(\frac{dv}{dx}\frac{Le}{2}\right) + H_3V_3 + H_4\left(\frac{dv}{dx}\frac{Le}{2}\right)$$

$$V(\xi) = H_1q_1 + H_2q_2\frac{L_e}{2} + H_3q_3 + H_4q_4\frac{L_e}{2}$$

We know $V = H\,q$

where

$$H = \begin{bmatrix} H_1 & H_2\frac{L_e}{2} & H_3 & H_4\frac{L_e}{2} \end{bmatrix}$$

Strain energy in the beam element we have

$$= \frac{1}{2}\int_0^L EI\,(d^2v/dx^2)^2\,dx$$

$$\frac{d^2v}{dx^2} = \frac{d}{dx}\left(\frac{dv}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{2}{L_e}\frac{dv}{d\xi}\right)$$

$$= \frac{2}{L_e}\frac{d}{dx}\left(\frac{dv}{d\xi}\right)$$

$$= \frac{2}{L_e}\frac{d}{dx}(m)$$

Where $m = \frac{dv}{d\xi}$

$$= \frac{2}{L_e}\left(\frac{2}{L_e}\frac{dm}{d\xi}\right)$$

$$\frac{d^2v}{dx^2} = \frac{4}{L_e^2}\left(\frac{d^2v}{d\xi^2}\right)$$

$$\left(\frac{d^2v}{dx^2}\right)^2 = \frac{16}{L_e^4}\left(\frac{d^2v}{d\xi^2}\right)^2$$

$$V = H\,q$$

$$\left(\frac{d^2v}{dx^2}\right)^2 = q^T \frac{16}{L_e^4}\left(\frac{d^2H}{d\xi^2}\right)^T\left(\frac{d^2H}{d\xi^2}\right) q$$

Where

$$\left(\frac{d^2H}{d\xi^2}\right) = \left[\frac{3\xi}{2} \ , \ \left(\frac{-1+3\xi}{2}\right)\frac{l_e}{2} \ , \ \frac{-3\xi}{2} \ , \ \left(\frac{1+3\xi}{2}\right)\frac{l_e}{2}\right]$$

Therefore total strain energy in a beam is

$$= \tfrac{1}{2}\int_e EI\,(d^2v/dx^2)^2\,dx$$

$$= \tfrac{1}{2}\int_e EI\,(d^2v/dx^2)^2\,l_e/2\;d\xi$$

$$= \frac{EI}{2}\frac{L_e}{2}\int q^T \frac{16}{L_e^4}\left(\frac{d^2H}{d\xi^2}\right)^T\left(\frac{d^2H}{d\xi^2}\right) q\;d\xi$$

Therefore total strain energy in a beam is

$$= \frac{1}{2} q^T \left\{ \frac{8EI}{L_e^3} \int_{-1}^{+1} \left(\frac{d^2H}{d\xi^2} \right)^T \left(\frac{d^2H}{d\xi^2} \right) \right\} q \, d\xi$$

$$= \frac{1}{2} q^T K q$$

Now taking the K component and integrating for limits -1 to +1 we get

$$K = \frac{EI}{Le^3} \begin{pmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{pmatrix}$$

K = Element stiffness matrix of the beam

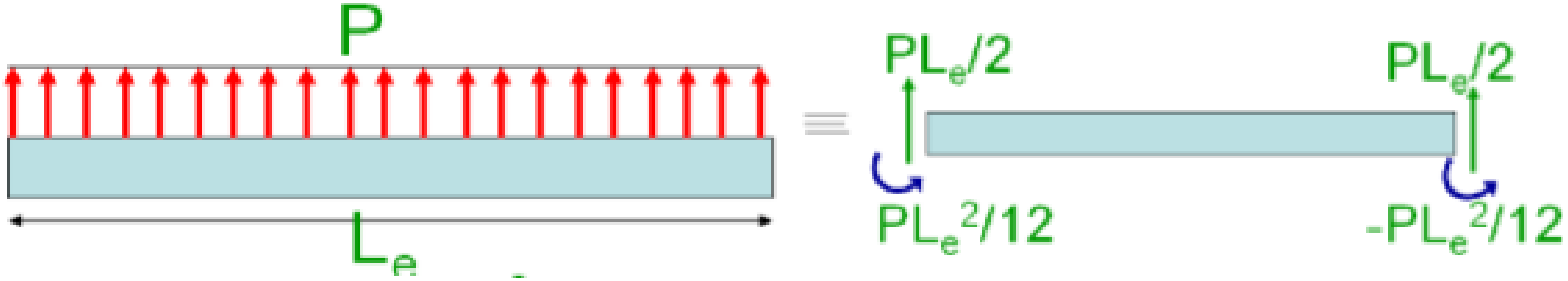
Finite element equillibriuam equation is

F = K Q

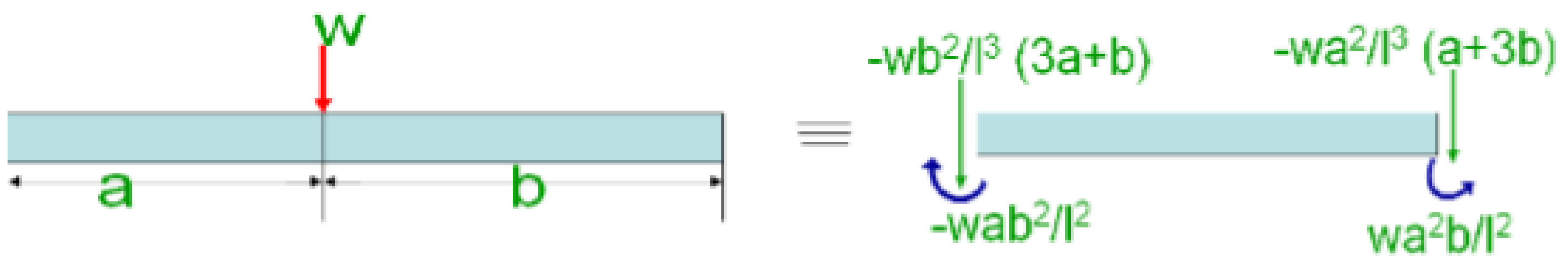
$$\begin{Bmatrix} F1y \\ m1 \\ F2y \\ m2 \end{Bmatrix} = \frac{EI}{Le^3} \begin{pmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{pmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ \bar{Q}_4 \end{Bmatrix}$$

Beam element forces with its equivalent loads

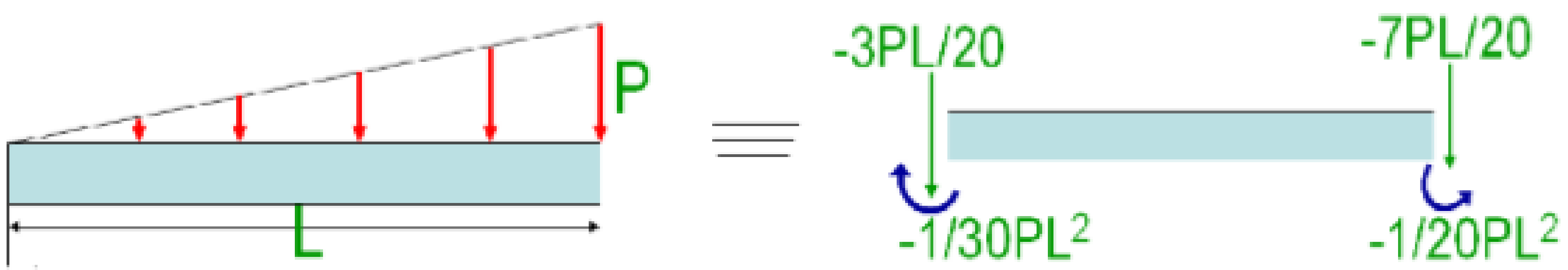
Uniformly distributed load



Point load on the element



Varying load



PROBLEM -1 FOR THE BEAM SHOWN IN FIGURE-1 DETERMINE THE DEFLECTION, SLOPE, BENDING MOMENT, SHEAR FORCE AND REACTIONS. GIVEN BEAM LENGTH 1m, BREATH 0.03m, DEPTH 0.04m AND $E = 2X10^{11}$ N/m^2.

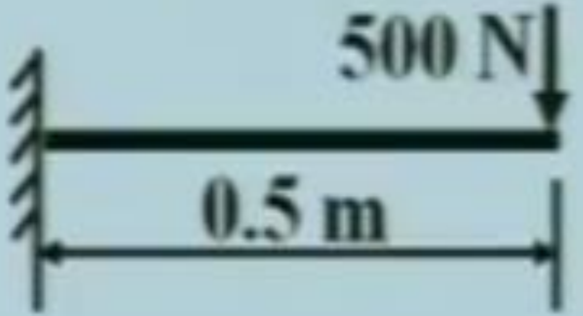


FIGURE -1

GIVEN CANTILEVER BEAM IS DISCRETIZED USING ONLY ONE ELEMENT AS SHOWN IN FIGURE -2

FIGURE -2

GIVEN DATA: $L = 0.5$ m

$b = 0.03$ m $d = 0.04$ m

$P = 500N$

$E = 2x10^{11} N/m^2$

$I = bd^3/12 = 1.60x10^{-7} m^4$

ANALYTICAL SOLUTION

$y_{max} = PL^3/3EI = 0.00065$ m $= 0.65$ mm

$(dy/dx)_{x=L} = PL^2/2EI = 0.00195$ rad

$(M_b)_{x=0} = PL = 250$ N-m

ELEMENTAL STIFFNESS IS GIVEN BY

$$[K_e] = \frac{EI}{l_e^3} \begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

$$\{f_e\} = P_c \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix}$$

SINCE THE BEAM IS DISCRETIZED BY ONLY ONE ELEMENT WE HAVE $[K]\{Q\} = \{F\}$

THEN $[K]\{Q\} = \{F\}$

$$\frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = P_c\begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix}$$

APPLYING THE BOUNDARY CONDITION WE HAVE Q_1 AND $Q_2 = 0$

THUS BY ELIMINATION APPROACH TO TREAT THE BOUNDARY CONDITION WE HAVE 1ST AND 2ND ROW AND COLUMNS ARE ELIMINATED.

THEN WE HAVE 2x2 MATRIX

THEN $[K]\{Q\} = \{F\}$

$$\frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = P_c\begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix}$$

$$\frac{EI}{L_e^3}\begin{bmatrix} 12 & -6l_e \\ -6l_e & 4l_e^2 \end{bmatrix}\begin{Bmatrix} Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} P_c \\ 0 \end{Bmatrix}$$ SOLVING WE HAVE

$$2Q_3 - L_eQ_4 = P_cL_e^3/6EI$$

$$-3Q_3 + 2L_eQ_4 = 0$$ THEN $Q_4 = (3/2L_e)Q_3$

$$2Q_3 - \cancel{L_e}(3/2\cancel{L_e})Q_3 = P_cL_e^3/6EI$$

THUS WE HAVE $Q_3 = P_c L_e^3/3EI$

SINCE $Q_4 = (3/2L_e)Q_3 = (3/2L_e)(P_c L_e^3/3EI)$

THUS $Q_4 = P_c L_e^2/2EI$

HENCE THE SOLUTION IS

$Q_1 = 0$, $Q_2 = 0$,

$Q_3 = P_c L_e^3/3EI = [(500 \times 0.5^3)/(3 \times 2X10^{11} \times 1.60 \times 10^{-7})]$

$= 0.00065$ m

$= 0.65$mm

$Q_4 = P_c L_e^2/2EI = [(500 \times 0.5^2)/(2 \times 2X10^{11} \times 1.60 \times 10^{-7})]$

$= 0.00195$ radians