

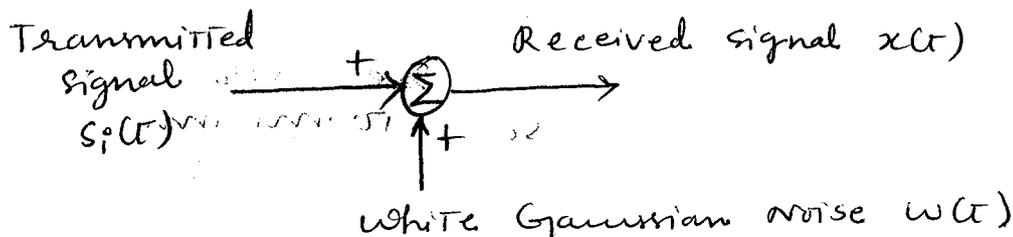
## Module-2 : Signaling over AWGN channels

- Additive white Gaussian noise (AWGN) is a basic noise model used in information theory to mimic the effect of many random processes that occur in nature.

- AWGN is often used as a channel model in which the only impairment to communication is a linear addition of wideband white noise with a constant spectral density.

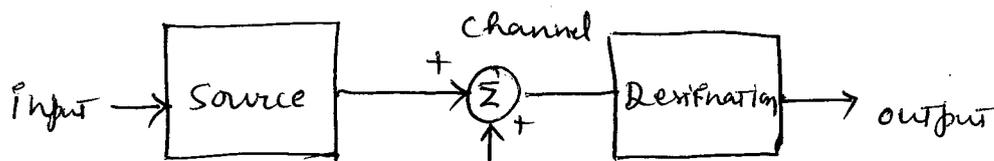
- The AWGN channel is a good model for many satellite and deep space communication links. AWGN is commonly used to simulate background noise of the channel under study.

- Fig@ shows AWGN model of a channel



Fig@ AWGN channel.

- Consider the transmission system (Communication System) consists of source and destination as shown in fig@



- The source output consists of a sequence of 1's and 0's, with each binary symbol being emitted every  $T_b$  seconds.

- The transmitting part of the digital communication system takes 1's and 0's emitted by the source and encodes them into distinct signals denoted by  $s_1(t)$  &  $s_2(t)$

- These signals i.e.  $s_1(t)$  and  $s_2(t)$  are suitable for transmission over the analog channel.
- Both  $s_1(t)$  and  $s_2(t)$  are real-valued energy signals denoted by

$$E_i = \int_0^{T_b} s_i^2(t) dt, \quad i = 1, 2 \quad - (1)$$

- With the analog channel represented by a AWGN as shown in fig (a). The received signal is defined by

$$x(t) = s_i(t) + w(t), \quad \begin{cases} 0 \leq t \leq T_b \\ i = 1, 2 \end{cases} \quad - (2)$$

where  $w(t)$  is the channel noise

- The receiver has the task of observing the received signal  $x(t)$  for duration of  $T_b$  seconds, and then making an estimate of the transmitted signal,  $s_i(t)$ .
- However, the presence of channel noise, the receiver will inevitably make occasional errors.
- To minimise errors at the receiver the requirement is to design the receiver so as to minimize the average probability of symbol error.
- The average probability of symbol error defined by

$$P_e = \pi_1 P(\hat{m} = 0 | 1 \text{ sent}) + \pi_2 P(\hat{m} = 1 | 0 \text{ sent})$$

\* Where  $\pi_1$  &  $\pi_2$  are the prior probability of the transmitting symbols 1 and 0 respectively, and  $\hat{m}$  is the estimate of the symbol 1 @ 0 sent by the source which is computed by the receiver.

\* The  $P(\hat{m} = 0 | 1 \text{ sent})$  and  $P(\hat{m} = 1 | 0 \text{ sent})$  are conditional probabilities.

- In minimizing the average probability between the receiver output and the symbol emitted by source, a reliable digital communication system should be developed.
- To achieve reliable communication system design objectives an  $M$ -ary alphabet should be used which are denoted by  $m_1, m_2, \dots, m_M$
- which involves how to choose the set of signals  $s_1(t), s_2(t), \dots, s_M(t)$  for representing symbols  $m_1, m_2, \dots, m_M$  respectively. It depends on geometric representation of signals.

## Geometric Representation of signals

- Geometric representation of signals provides a compact alternative characterization of signals and simplify analysis of their performance as modulation signals.
- The essence of geometric representation of signals is to represent any set of  $M$  energy signals  $\{s_i(t)\}$  as linear combination of ' $N$ ' orthonormal basis function, where  $N \leq M$ .
- The message which are transmitted over the carrier from a signal space  $\textcircled{a}$  vector space. From such representation we can study
  - i) probability of error in transmission
  - ii) Distance  $\textcircled{a}$  separation between individual message.
- The vector space representation is the characteristics of every digital modulation technique.

- let there be 'M' number of energy signals which forms the input signal set i.e

$$S_i(t) = \{s_1(t), s_2(t), \dots, s_M(t)\} \quad - (1)$$

- let these signals be represented in terms of 'N' number of orthonormal basis functions i.e

$$\phi_j(t) = \{\phi_1(t), \phi_2(t), \dots, \phi_N(t)\} \quad - (2)$$

- Then the linear relationship between  $s_i(t)$  and  $\phi_j(t)$  can be written as

$$s_i(t) = s_{i1} \phi_1(t) + s_{i2} \phi_2(t) + \dots + s_{iN} \phi_N(t) \quad - (3)$$

- For real-valued energy signals of  $s_1(t), s_2(t), \dots, s_M(t)$  each of duration 'T' seconds can be written as

$$s_i(t) = \sum_{j=1}^N s_{ij} \phi_j(t), \quad \begin{cases} 0 \leq t \leq T \\ i = 1, 2, \dots, M \end{cases} \quad - (4)$$

- In the above equation (4)  $s_{ij}$  are called co-efficients of the expansion and they are defined by

$$s_{ij} = \int_0^T s_i(t) \phi_j(t) dt \quad \begin{cases} i = 1, 2, \dots, M \\ j = 1, 2, \dots, N \end{cases} \quad - (5)$$

Here 'T' is the duration of the symbol  $s_i(t)$ .

- Since the basis function  $\phi_1(t), \phi_2(t), \dots, \phi_N(t)$  are orthonormal they satisfy following property

$$\int_0^T \phi_i(t) \phi_j(t) dt = \delta_{ij} = \begin{cases} 1 & \text{for } i=j \\ 0 & \text{for } i \neq j \end{cases} \quad - (6)$$

where  $\delta_{ij}$  is the Kronecker delta.

- when  $i=j$  above equation will be

$$\int_0^T \phi_i(t) \phi_i(t) dt = \int_0^T \phi_i^2(t) dt = 1 \quad \text{as per eq (6)}$$

- Thus from Equation (6) the first condition states that each basis function is normalized to have unit energy i.e.  $K=1$  when  $i=j$

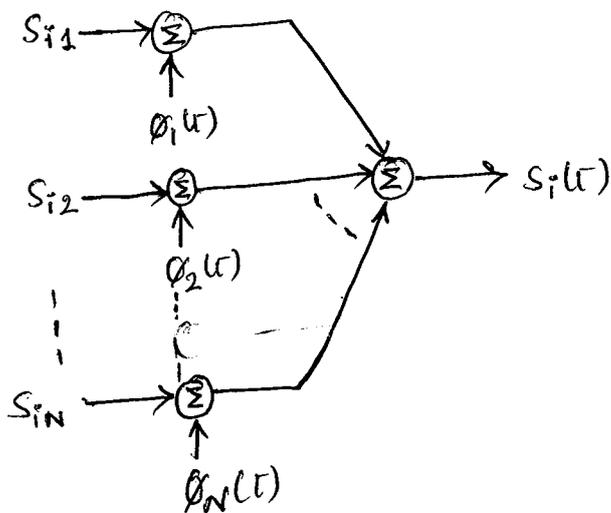
- The second condition from Equation (6) i.e. ( $i \neq j$ ) states that the basis function  $\phi_1(t), \phi_2(t) \dots \phi_N(t)$  are orthogonal with respect to each other over the interval  $0 \leq t \leq T$ .

$$\text{i.e. } \int_0^T \phi_i(t) \phi_j(t) = 0 \quad \text{when } i \neq j$$

### Generating the signal $s_i(t)$ (synthesizer)

- Given the 'N' elements of the vector  $s_i$  operating as input we may use below figure (synthesizer) to generate the signal  $s_i(t)$ , which follows the Equation

$$s_i(t) = \sum_{j=1}^N s_{ij} \phi_j(t)$$



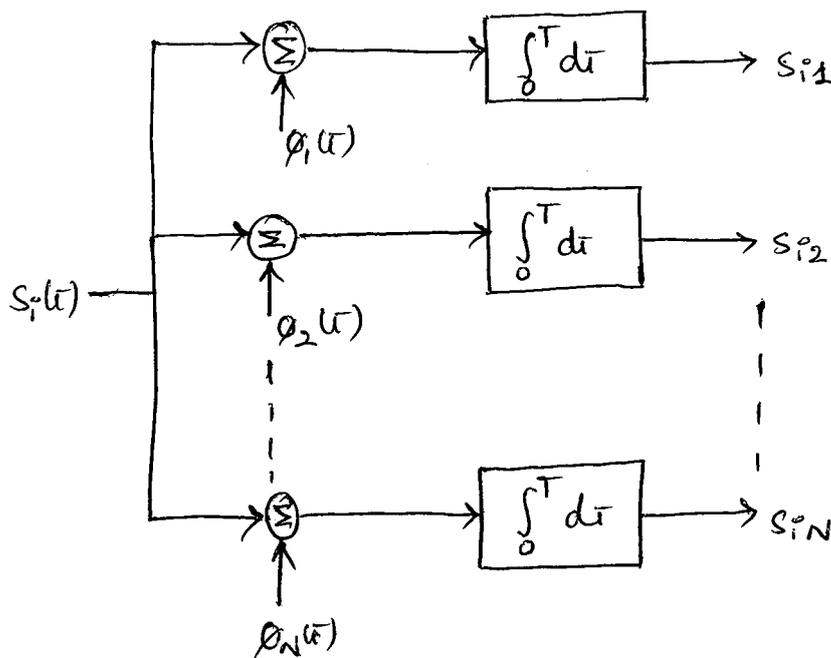
- The above figure consists of a bank of 'N' multipliers with each multiplier having its own basis function followed by a summer.

- The scheme of the above figure may be viewed as synthesizer (generating and combining signals of different frequencies)

## Reconstructing the signal vector $\{s_i\}$ (Analyzer)

- Given the signals  $s_i(t)$ ,  $i = 1, 2 \dots N$  operating as input we may use the scheme shown in figure. 10 calculate the Co-efficients  $s_{i1}, s_{i2} \dots s_{iN}$  which follows the Equation ie

$$s_{ij} = \int_0^T s_i(t) \phi_j(t) dt.$$



- The reconstruction scheme consists of a bank of 'N' product integrators @ correlator with common input and with each one of them supplied with its own basis function
- The above scheme may be viewed as an analyzer (Device that analyses given data)

## An Example of Two-Dimensional Signal Space with Three Symbols

Let us consider the vector space representation of  $M=3$  (message symbols) with the help of  $N=2$  (orthonormal basis) functions.

$$\text{w.k.t } s_i(t) = \sum_{j=1}^N s_{ij} \phi_j(t) \quad \text{--- (1)} \quad \begin{array}{l} i = 1, 2, 3 \dots M \\ j = 1, 2, 3 \dots N \end{array}$$

for  $N=2$  and  $M=3$  eq (1) can be

rewritten as

$$s_i(t) = \sum_{j=1}^2 s_{ij} \phi_j(t)$$

$$= s_{i1}(t) \phi_1(t) + s_{i2}(t) \phi_2(t)$$

$$s_1(t) = s_{11}(t) \phi_1(t) + s_{12}(t) \phi_2(t)$$

$$s_2(t) = s_{21}(t) \phi_1(t) + s_{22}(t) \phi_2(t)$$

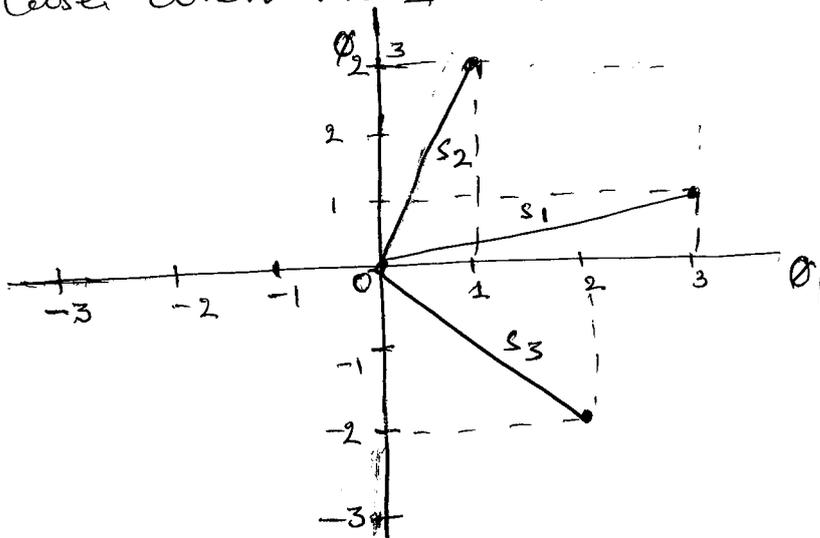
$$s_3(t) = s_{31}(t) \phi_1(t) + s_{32}(t) \phi_2(t)$$

$$\begin{array}{l} i = 1, 2, 3 \dots M \\ j = 1, 2 \dots N \end{array}$$

$$\begin{array}{l} M \\ N \leq M \end{array}$$

- We may visualize the set of signal vectors  $\{s_i | i = 1, 2 \dots M\}$  as defining a corresponding set of  $M$  points in an  $N$ -dimensional Euclidean space with  $N$  mutually perpendicular axes labeled  $\phi_1, \phi_2 \dots \phi_N$ . This  $N$ -dimensional Euclidean space is called the signal space.

- Fig shows geometric representation of signals for the case when  $N=2$  and  $M=3$ .



- From figure it shows three message vectors  $s_1, s_2$  and  $s_3$ . The position of these vectors depends upon the co-efficients  $s_{11}, s_{12}, s_{21}, s_{22}$  etc
- From figure  $\phi_1$  and  $\phi_2$  are perpendicular to each other since they are orthogonal.
- The  $\phi_1 - \phi_2$  signal space is called Euclidean space.
- The similar concept can be extended to visualizing a set of  $N$ -dimensional Euclidean space. It provides the mathematical basis for the geometric representation of energy signals in a conceptually satisfying manner. For  $N$ -dimensional Euclidean space there will be  $\phi_1, \phi_2, \dots, \phi_N$  mutually perpendicular axes.

### Absolute value @ norm of a vector

- In an  $N$ -dimensional Euclidean space we may define lengths of vector and angles between vectors.
- Consider the vector  $s_i$  which is completely determined by its co-efficients i.e.

$$s_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix} \quad \text{and } i = 1, 2, \dots, M \quad \text{--- (1)}$$

- The absolute value of a signal vector  $s_i$  is given by the symbol  $\|s_i\|$ .
- The squared length of any signal vector  $s_i$  is defined to be the inner product @ dot product of  $s_i$ 's given by

$$\begin{aligned} \|s_i\|^2 &= s_i^T s_i \quad \text{--- (2)} \\ &= [s_{i1}, s_{i2}, s_{i3}, \dots, s_{iN}] \begin{bmatrix} s_{i1} \\ s_{i2} \\ s_{i3} \\ \vdots \\ s_{iN} \end{bmatrix} \end{aligned}$$

$$= s_{i1}^2 + s_{i2}^2 + s_{i3}^2 + \dots + s_{iN}^2$$

$$= \sum_{j=1}^N s_{ij}^2 \quad \text{and } i=1, 2, \dots, M \quad \text{--- (3)}$$

where  $s_{ij}$  is the  $j^{\text{th}}$  element of  $s_i$  and the subscript 'T' denotes matrix representation.

### Relationship between signal Energy and its vector

- we know that, by definition the energy of a signal  $s_i(t)$  of duration 'T' seconds is

$$E_i = \int_0^T s_i^2(t) dt \quad i=1, 2, \dots, M. \quad \text{--- (1)}$$

$$\text{w.k.T } s_i(t) = \sum_{j=1}^N s_{ij} \phi_j(t) \quad \text{--- (2)}$$

Substitute eq (2) in Equation (1)

$$E_i = \int_0^T \left[ \sum_{j=1}^N s_{ij} \phi_j(t) \right] \left[ \sum_{k=1}^N s_{ik} \phi_k(t) \right] dt \quad \text{--- (3)}$$

Interchanging the order of summation and Integration which can do because they are both linear operations and then rearranging terms we get

$$E_i = \sum_{j=1}^N \sum_{k=1}^N s_{ij} s_{ik} \int_0^T \phi_j(t) \phi_k(t) dt \quad \text{--- (4)}$$

$$\text{we know that } \int_0^T \phi_j(t) \phi_k(t) dt = \begin{cases} 1 & \text{for } j=k \\ 0 & \text{for } j \neq k \text{ (orthogonal)} \end{cases}$$

From the above conditions eq (4) can be written as

$$E_i = \begin{cases} \sum_{j=1}^N \sum_{k=1}^N s_{ij} s_{ik} & \text{if } j=k \\ 0 & \text{if } j \neq k \end{cases}$$

$$E_i = \sum_{j=1}^N \sum_{k=1}^N s_{ij} \cdot s_{ik} \quad \text{with } j=k \quad \text{--- (5)}$$

- Hence both the summation represents same index and hence above Equation (4) becomes

$$E_i = \sum_{j=1}^N s_{ij}^2 = \|s_i\|^2 \quad - (6)$$

$$\boxed{E_i = \|s_i\|^2} \quad - (7)$$

- Thus from Equation (7) shows that the signal energy is equal to squared length of the signal vector  $s_i(t)$ .

- In case of a pair of signals  $s_i(t)$  and  $s_k(t)$  represented by the signal vectors  $s_i$  and  $s_k$  respectively.

$$E_i = \int_0^T s_i(t) s_k(t) dt = s_i^T s_k$$

$$\boxed{E_i = s_i^T s_k} \quad - (8)$$

Equation (8) states that the inner product of the energy signals  $s_i(t)$  and  $s_k(t)$  over the interval  $[0, T]$  is equal to the inner product of their respective vectors representations  $s_i$  and  $s_k$ .

### Euclidean Distance

- The Euclidean distance between two signals vectors is given as

$$d_{ik} = \|s_i - s_k\| \quad - (9)$$

where  $\|s_i - s_k\|$  is the Euclidean distance  $d_{ik}$  between two points represented by the signal vectors  $s_i$  and  $s_k$ .

- The squared Euclidean distance  $s_i(t)$  and  $s_k(t)$  is described by

$$\|s_i - s_k\|^2 = \sum_{j=1}^N (s_{ij} - s_{kj})^2 \quad - (10)$$

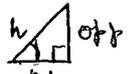
Since  $E_i = \int_0^T s_i^2(t) dt = \|s_i\|^2$  from this basis the above Equation (10) can be written as

$$\|s_i - s_k\|^2 = \int_0^T [s_i(t) - s_k(t)]^2 dt \quad \text{--- (3)}$$

- To complete the geometric representation of energy signals we need to have a representation for the angle  $\theta_{ik}$  between two signal vectors  $s_i$  &  $s_k$ .

- By definition of the cosine of the angle  $\theta_{ik}$  is equal to the inner product of these two vectors divided by the product of their individual norms given by

$$\cos(\theta_{ik}) = \frac{s_i^T s_k}{\|s_i\| \|s_k\|} \quad \text{--- (4)}$$



$$\cos \theta = \frac{\text{adj}}{\text{hyp}}$$

$$\text{hyp}^2 = \text{adj}^2 + \text{opp}^2$$

- The two vectors  $s_i$  and  $s_k$  are thus orthogonal (or) perpendicular to each other if their inner product  $s_i^T s_k = 0$  in which case  $\theta_{ik} = 90^\circ$

### Gram-Schmidt orthogonalization procedure

- In mathematics, particularly linear algebra and numerical analysis, the Gram-Schmidt process is a method for orthonormalising a set of vectors in an inner product space.

- We know that any signal vector can be represented in terms of orthonormal basis functions  $\phi_1(t), \phi_2(t), \dots, \phi_N(t)$ . Gram-Schmidt orthogonalization procedure is the tool to obtain the orthonormal basis function  $\phi_i(t)$ .

To derive an expression for  $\phi_i(t)$

- Suppose we have set of 'M' energy signals denoted by  $s_1(t), s_2(t), \dots, s_M(t)$ .

- Starting with  $s_1(t)$  chosen from set arbitrarily the basis function is defined by

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} \quad \text{--- (1)} \quad \text{where } E_1 \text{ is the energy of the signal } s_1(t).$$

- From Equation ① we can write

$$s_1(t) = \sqrt{E_1} \phi_1(t) \quad \text{--- ②}$$

We know that  $\sum_{j=1}^N s_{ij} \phi_j(t)$  for  $N=1$  eq ② can be

written as

$$s_1(t) = s_{11} \phi_1(t) \quad \text{--- ③}$$

- From the above Equation ③ we obtain  $s_{11} = \sqrt{E_1}$  and  $\phi_1(t)$  has unit energy.

- Next, using the signal  $s_2(t)$  we define the coefficient  $s_{21}$  as

$$s_{21} = \int_0^T s_2(t) \phi_1(t) dt \quad \text{--- ④}$$

- Let  $g_2(t)$  be a new intermediate function which is given as

$$g_2(t) = s_2(t) - s_{21} \phi_1(t) \quad \text{--- ⑤}$$

- The function is orthogonal to  $\phi_1(t)$  over the interval 0 to T.

- The second function which is given as

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{E_{g_2}}} \quad \text{--- ⑥}$$

$E_{g_2} = \int_0^T g_2^2(t) dt$  is the energy of  $g_2(t)$

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t) dt}} \quad \text{--- ⑦}$$

② To prove that  $\phi_2(t)$  has unit energy

- Energy of  $\phi_2(t)$  will be  $\int_0^T \phi_2^2(t) dt$

$$= \int_0^T \left[ \frac{g_2(t)}{\sqrt{E_{g_2}}} \right]^2 dt$$

$$= \frac{1}{E_{g_2}} \int_0^T g_2^2(t) dt$$

$$= \frac{1}{E_{g_2}} \int_0^T g_2^2(t) dt$$

$$\text{w.k.T } E_{g_2} = \int_0^T g_2^2(t) dt$$

$$= \frac{1}{E_{g_2}} \times E_{g_2}$$

$$\int_0^T \phi_2^2(t) dt = 1$$

⑥ To prove that  $\phi_1(t)$  and  $\phi_2(t)$  are orthogonal

$$\text{Consider } \int_0^T \phi_1(t) \phi_2(t) dt$$

Substitute the values of  $\phi_1(t)$  and  $\phi_2(t)$  in the above equation i.e

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} \quad \text{and} \quad \phi_2(t) = \frac{g_2(t)}{\sqrt{E_{g_2}}}$$

$$= \int_0^T \frac{s_1(t)}{\sqrt{E_1}} \frac{g_2(t)}{\sqrt{E_{g_2}}} dt$$

$$\int_0^T \phi_1(t) \phi_2(t) dt = \frac{1}{\sqrt{E_1} \sqrt{E_{g_2}}} \int_0^T s_1(t) g_2(t) dt$$

$$\text{Substitute } g_2(t) = [s_2(t) - s_{2,1} \phi_1(t)]$$

$$= \frac{1}{\sqrt{E_1} \sqrt{E_{g_2}}} \int_0^T s_1(t) [s_2(t) - s_{2,1} \phi_1(t)] dt$$

$$= \frac{1}{\sqrt{E_1} \sqrt{E_{g_2}}} \left[ \int_0^T s_1(t) s_2(t) dt - \int_0^T s_{2,1} s_1(t) \phi_1(t) dt \right]$$

$$\text{w.k.T } s_{2,1} = \int_0^T s_2(t) \phi_1(t)$$

$$\int_0^T \phi_1(t) \phi_2(t) dt = \frac{1}{\sqrt{E_1} \sqrt{E_{g_2}}} \left[ \int_0^T s_1(t) s_2(t) dt - \int_0^T \int_0^T s_2(t) \phi_1(t) s_1(t) \phi_1(t) dt \right]$$

- From the given Equation there is a product of two terms  $s_1(t)$  and  $s_2(t)$ , but the two symbols are not present at a time. Hence the product of  $s_1(t)$  and  $s_2(t)$  i.e.  $s_1(t)s_2(t) = 0$  and hence the integration terms in RHS will be zero. i.e.

$$\int_0^T \phi_1(t) \phi_2(t) dt = 0$$
 Thus the two basis functions are orthonormal.

### Generalised Equation for orthonormal basis functions

- The generalised Equation for orthonormal basis function can be written by considering the following Equation i.e.

$$\phi_i(t) = \frac{g_i(t)}{\sqrt{E_{g_i}}}, \quad i = 1, 2, 3, \dots, N$$

where  $g_i(t)$  is given by the generalised Equation

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij} \phi_j(t)$$

Note  
( $i-1$ )  $\rightarrow$   $s_i(t)$  all ready taken consideration

where the Co-efficients

$$s_{ij} = \int_0^T s_i(t) \phi_j(t) dt \quad j = 1, 2, \dots, i-1$$

For  $i=1$  the  $g_i(t)$  reduces to  $s_i(t)$ .

Given the  $g_i(t)$ , we may define the set of basis function

$$\phi_j(t) = \frac{g_j(t)}{\sqrt{\int_0^T g_j^2(t) dt}} \quad j = 1, 2, \dots, N$$

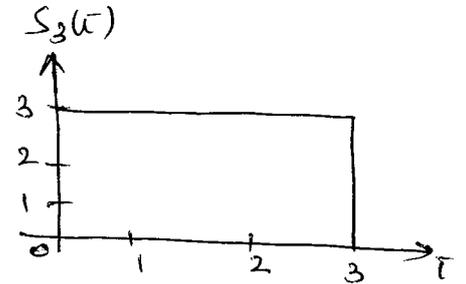
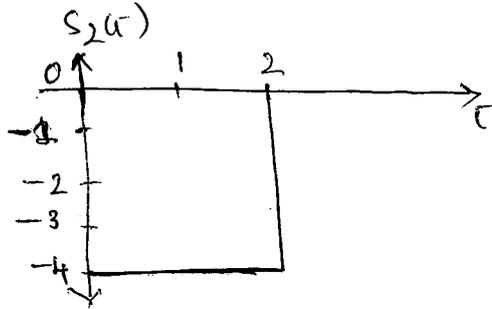
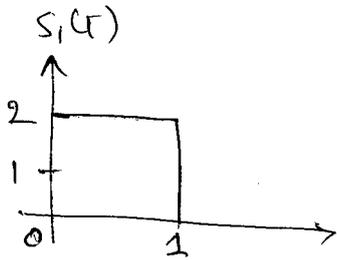
which form an orthonormal set. The dimension 'N' is less than  
 (a) Equal to the number of given signals, M depending on one of two possibilities.

- \* The signals  $s_1(t), s_2(t), \dots, s_M(t)$  form a linearly independent set, in which case  $N = M$
- \* The signals  $s_1(t), s_2(t), \dots, s_M(t)$  are not linearly independent

# Problems on Gram-Schmidt orthogonalization

## Problem - 1

Using the Gram-Schmidt orthogonalization procedure, find a set of orthonormal basis functions represent the three signals  $s_1(t)$ ,  $s_2(t)$  &  $s_3(t)$  as shown in figure.



Soln

All the three signals  $s_1(t)$ ,  $s_2(t)$  and  $s_3(t)$  are not linear combination of each other, hence they are linearly independent. Hence we require three basis functions.

To obtain  $\phi_1(t)$

Energy of  $s_1(t)$ ,  $E_1 = \int_0^T s_1^2(t) dt$

$$E_1 = \int_0^1 (2)^2 dt = 4[1-0]$$

$$E_1 = 4$$

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{2}{\sqrt{4}} = \frac{2}{2} = 1$$

$$\phi_1(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

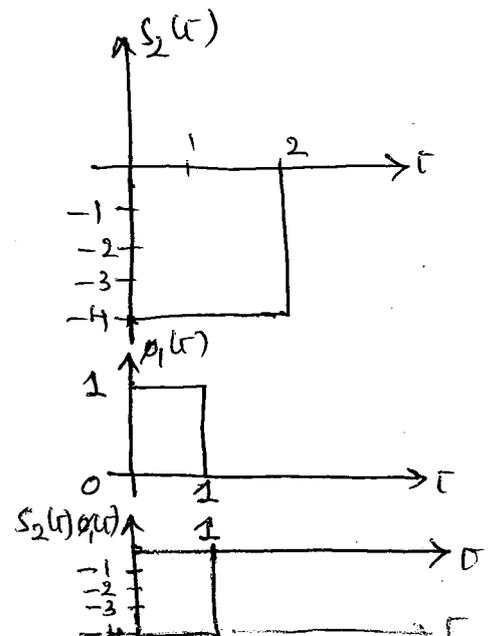
To obtain  $\phi_2(t)$

$$g_2(t) = s_2(t) - s_{21}\phi_1(t)$$

$$s_{21} = \int_0^T s_2(t)\phi_1(t) dt$$

$$s_{21} = \int_0^1 (-4)(1) dt$$

$$s_{21} = \int_0^1 -4 \quad 0 \leq t \leq 1$$



$$g_2(t) = s_2(t) - s_{21}\phi_1(t)$$

$$s_{21}\phi_1(t) = \begin{cases} -4 & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$g_2(t) = \begin{cases} -4 & \text{for } 1 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{E_{g_2}}}$$

$$E_{g_2} = \int_0^T g_2^2(t) dt$$

$$E_{g_2} = \int_1^2 (-4)^2 dt$$

$$E_{g_2} = 16(2-1)$$

$$\boxed{E_{g_2} = 16}$$

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{E_{g_2}}} = \frac{-4}{\sqrt{16}} = \frac{-4}{4}$$

$$\phi_2(t) = \begin{cases} -1 & \text{for } 1 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

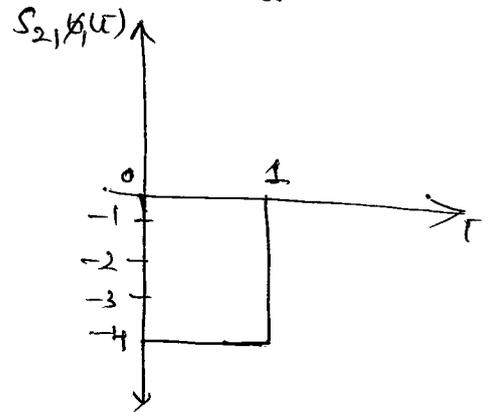
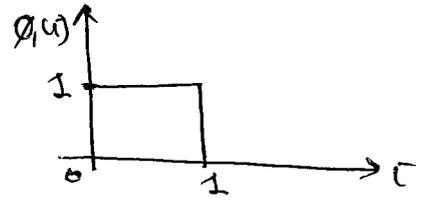
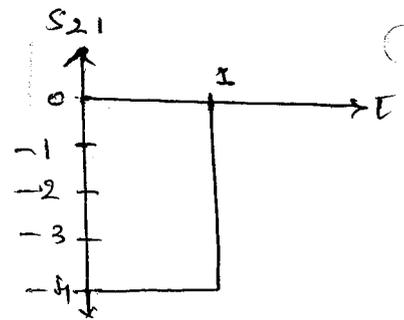
To obtain  $\phi_3(t)$

$$g_1(t) = s_1(t) - \sum_{j=1}^{i-1} s_{1j}\phi_j(t)$$

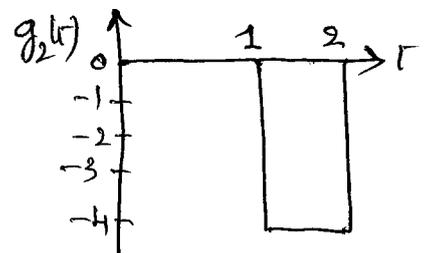
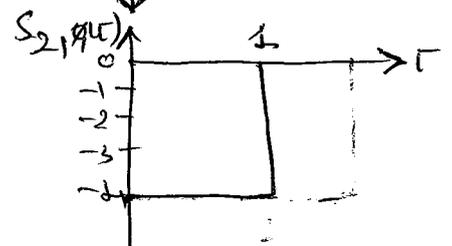
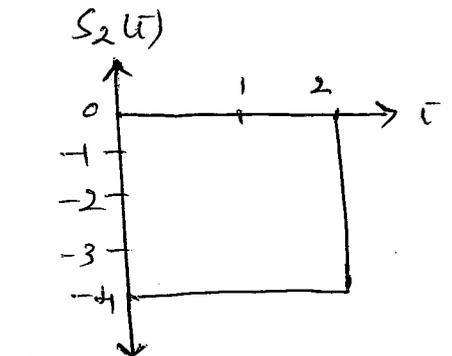
$$g_3(t) = s_3(t) - s_{31}\phi_1(t) - s_{32}\phi_2(t)$$

$$s_{31} = \int_0^T s_3(t)\phi_1(t) dt$$

$$s_{31}\phi_1(t) = \begin{cases} 3 & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

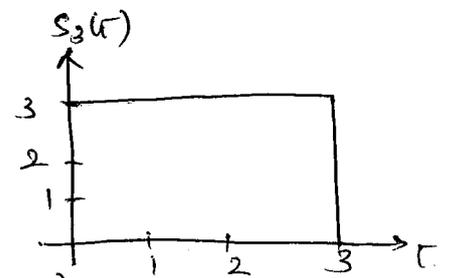


(c)



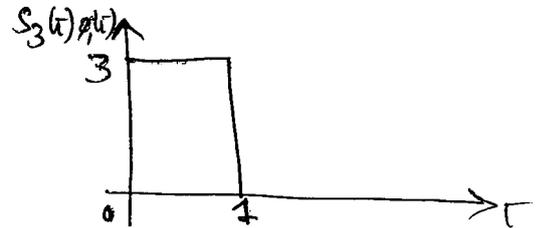
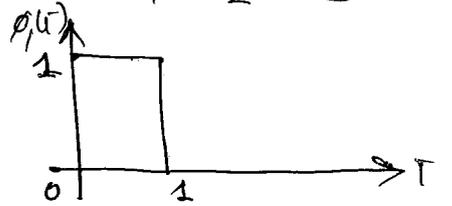
$$S_{31} = \int_0^1 (3)(1) dt$$

$$s_{31} = \begin{cases} 3 & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



$$S_{32} = \int_0^T s_3(t) \phi_2(t) dt$$

$$s_3(t) \phi_2(t) = \begin{cases} -3 & 1 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

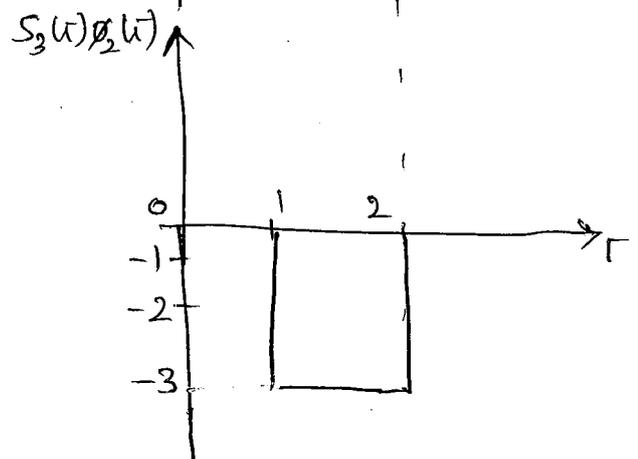
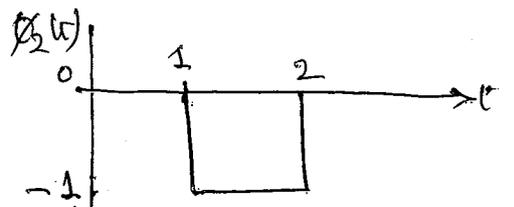
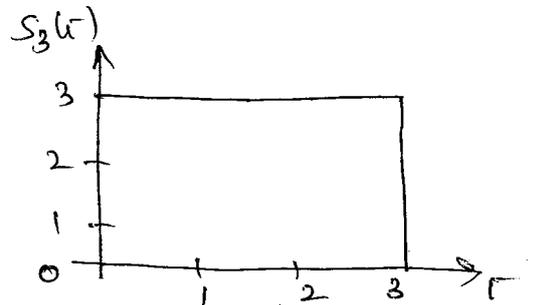


$$S_{32} = \int_1^2 (-3)(-1) dt$$

$$S_{32} = -3 [2-1]$$

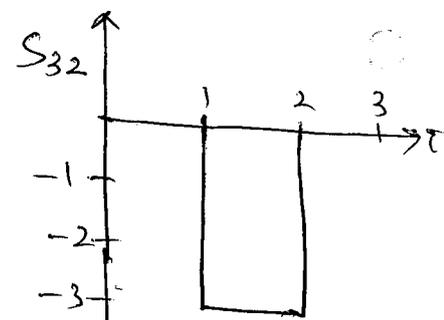
$$S_{32} = -3(1)$$

$$s_{32} = \begin{cases} -3 & 1 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

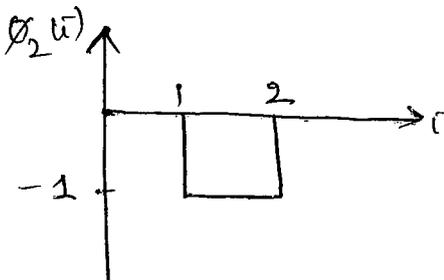


\* continued in next page

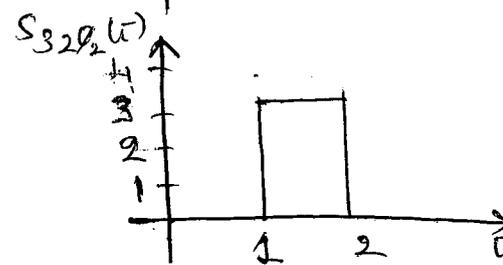
$$s_{32}\phi_2(t) = \begin{cases} 3 & 1 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$



$$g_3(t) = (s_3(t) - s_{31}\phi_1(t)) - s_{32}\phi_2(t)$$



$$g_3(t) = \begin{cases} 3 & \text{for } 2 \leq t \leq 3 \\ 0 & \text{otherwise} \end{cases}$$



$$E_{g_3} = \int_0^T g_3^2(t) dt$$

$$E_{g_3} = \int_2^3 (3)^2 dt$$

$$E_{g_3} = 9 [3-2]$$

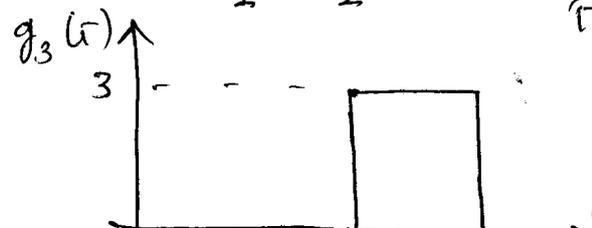
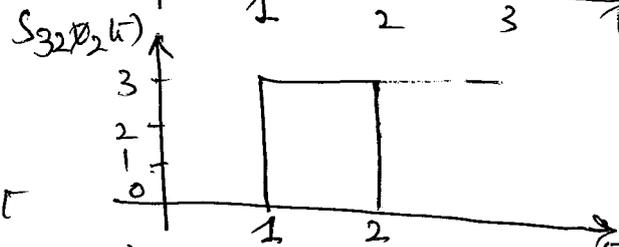
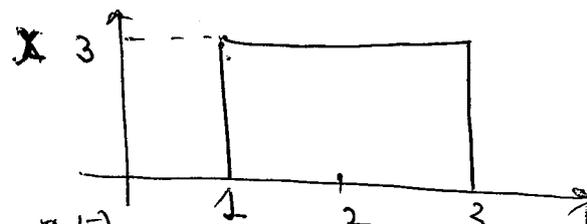
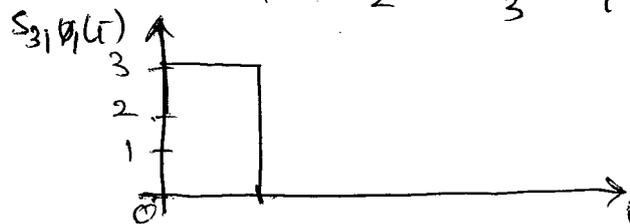
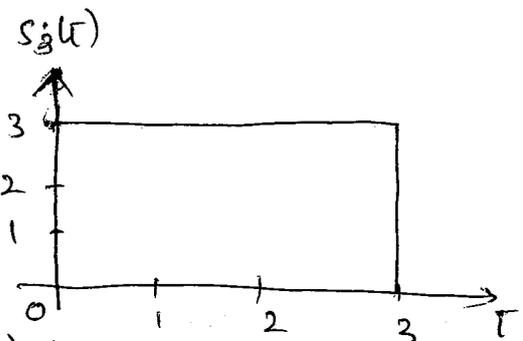
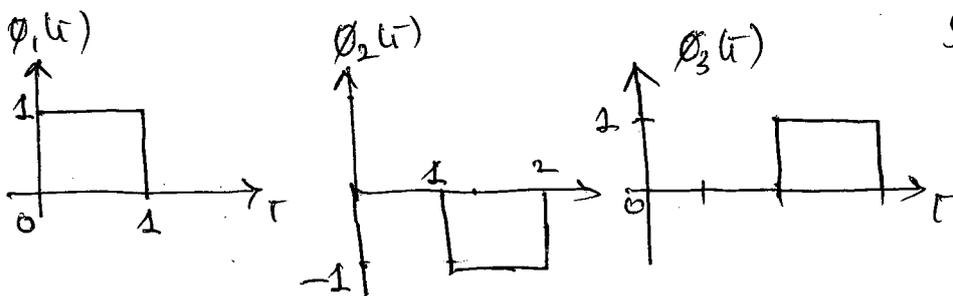
$$E_{g_3} = 9$$

$$\phi_3(t) = \frac{g_3(t)}{\sqrt{E_{g_3}}}$$

$$\phi_3(t) = \frac{3}{\sqrt{9}} = 1$$

$$\phi_3(t) = \begin{cases} 1 & 2 \leq t \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Basis functions as shown below



## Problem on Gram-Schmidt Orthogonalisation

Problem - (2)

Consider the signals  $s_1(t)$ ,  $s_2(t)$ ,  $s_3(t)$  and  $s_4(t)$  as given below. Find an orthonormal basis for these set of signals using Gram-Schmidt orthogonalisation procedure.

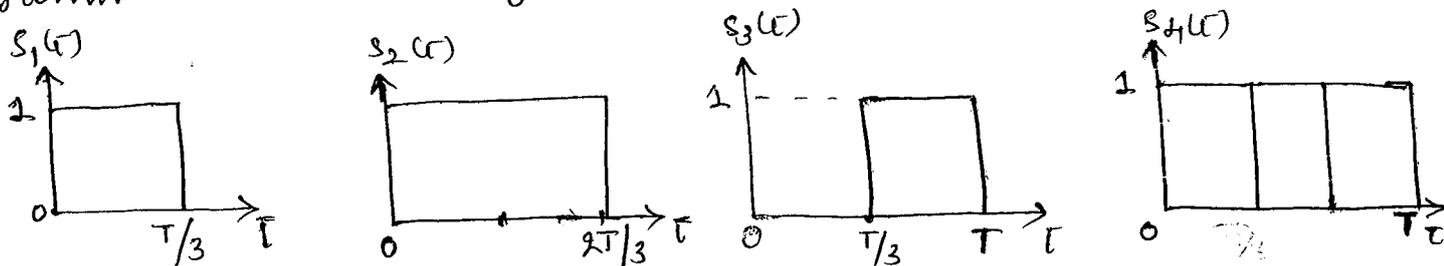


Fig: sketch of  $s_1(t)$ ,  $s_2(t)$ ,  $s_3(t)$  and  $s_4(t)$

Soln: From the above figures  $s_4(t) = s_1(t) + s_3(t)$ , This means all four signals are not linearly independent.

Gram-Schmidt orthogonalisation procedure is carried out for a subset which is linearly independent. Here  $s_1(t)$ ,  $s_2(t)$  and  $s_3(t)$  are linearly independent. Hence we will determine orthonormal.

To obtain  $\phi_1(t)$

$$\begin{aligned} \text{Energy of } s_1(t) \text{ is } E_1 &= \int_0^T s_1^2(t) dt \\ &= \int_0^{T/3} (1)^2 dt = [t]_0^{T/3} = [T/3 - 0] \end{aligned}$$

$$E_1 = T/3$$

$$\text{W.K.T } \phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \begin{cases} \frac{1}{\sqrt{T/3}} & \text{for } 0 \leq t \leq T/3 \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_1(t) = \begin{cases} \sqrt{3/T} & \text{for } 0 \leq t \leq T/3 \\ 0 & \text{otherwise.} \end{cases}$$

To obtain  $\phi_2(t)$

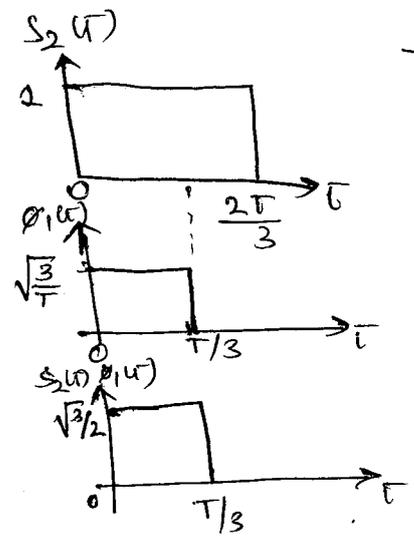
w.k.T  $S_{21} = \int_0^T S_2(t) \phi_1(t) dt$

$$S_{21} = \int_0^{T/3} (1) \sqrt{3/T} dt$$

$$= \left( \sqrt{\frac{3}{T}} \right) \times \left( \frac{T}{3} \right)$$

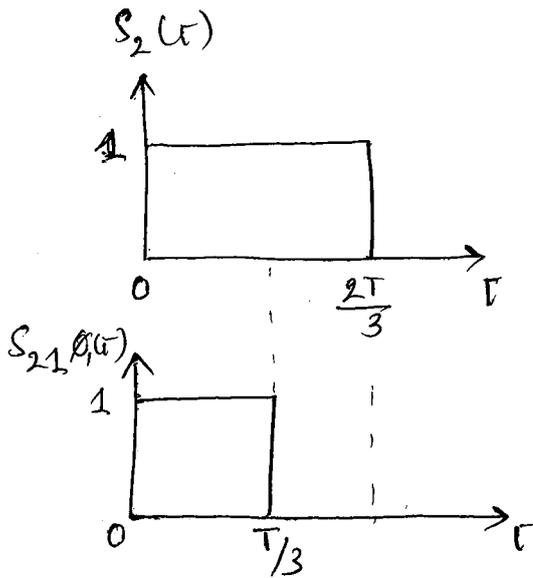
$$= \left( \frac{\sqrt{3}}{\sqrt{T}} \times \frac{\sqrt{T} \sqrt{T}}{\sqrt{3} \sqrt{3}} \right) = \frac{\sqrt{T}}{\sqrt{3}}$$

$$S_{21} = \sqrt{\frac{T}{3}}$$



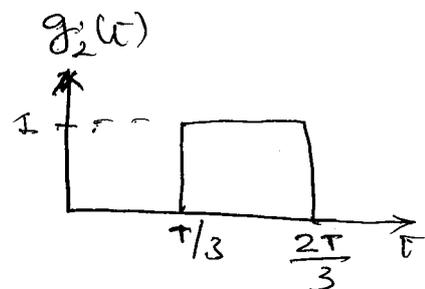
$$S_{21} \phi_1(t) = \begin{cases} \sqrt{T/3} \times \sqrt{3/T} & \text{for } 0 \leq t \leq T/3 \\ 0 & \text{otherwise} \end{cases}$$

$$S_{21} \phi_1(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq T/3 \\ 0 & \text{otherwise} \end{cases}$$



- From the above figures the intermediate function can be defined as

$$g_2(t) = S_2(t) - S_{21} \phi_1(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq T/3 \\ 1 & \text{for } T/3 \leq t \leq 2T/3 \\ 0 & \text{for } t > 2T/3 \end{cases}$$



Energy of  $g_2(t)$  will be

$$E_{g_2} = \int_0^T g_2^2(t) dt = \int_{T/3}^{2T/3} (1)^2 dt$$

$$= \left[ t \right]_{T/3}^{2T/3} = \left[ \frac{2T}{3} - \frac{T}{3} \right]$$

$$E_{g_2} = T/3$$

Now  $\phi_2(t) = \frac{g_2(t)}{\sqrt{E_{g_2}}} = \frac{1}{\sqrt{T/3}} = \sqrt{3/T}$

$$\phi_2(t) = \begin{cases} \sqrt{3/T} & \text{for } T/3 \leq t \leq 2T/3 \\ 0 & \text{otherwise} \end{cases}$$

To obtain  $\phi_3(t)$

We know that the generalised equation for Gram-Schmidt procedure

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij} \phi_j(t) \quad i = 1, 2, \dots, N$$

with  $N=3$

$$g_3(t) = s_3(t) - \sum_{j=1}^2 s_{3j} \phi_j(t)$$

$$= s_3(t) - [s_{31} \phi_1(t) + s_{32} \phi_2(t)]$$

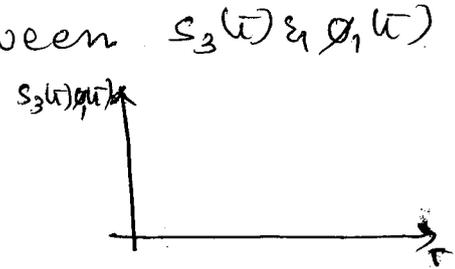
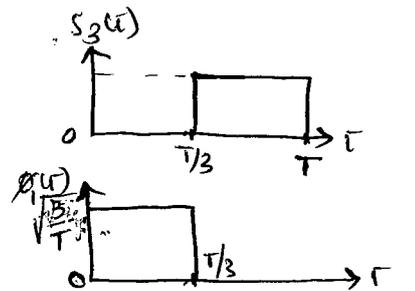
$$g_3(t) = s_3(t) - s_{31} \phi_1(t) + s_{32} \phi_2(t)$$

w.k.t  $s_{ij} = \int_0^T s_i(t) \phi_j(t) dt$

$$s_{31} = \int_0^T s_3(t) \phi_1(t) dt = 0$$

Since there is no overlap between  $s_3(t)$  &  $\phi_1(t)$

$$s_{32} = \int_0^T s_3(t) \phi_2(t) dt$$



$$S_{32} = \int_{T/3}^{2T/3} s_3(t) \phi_2(t) dt$$

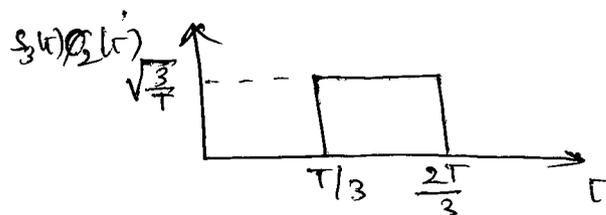
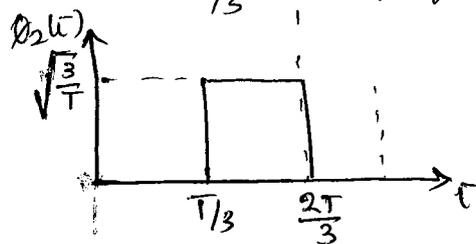
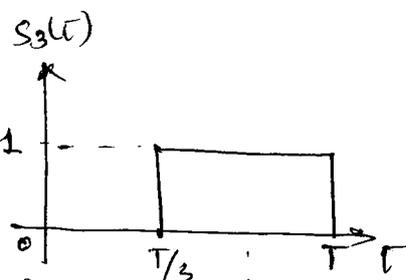
$$S_{32} = \int_{T/3}^{2T/3} (1) \times \sqrt{\frac{3}{T}} dt$$

$$= \left[ \sqrt{\frac{3}{T}} \times t \right]_{T/3}^{2T/3}$$

$$= \sqrt{\frac{3}{T}} \times \left[ \frac{2T}{3} - \frac{T}{3} \right]$$

$$= \sqrt{\frac{3}{T}} \cdot \frac{T}{3} = \sqrt{\frac{3}{T}} \sqrt{\frac{T}{3}} \sqrt{\frac{T}{3}}$$

$$S_{32} = \sqrt{\frac{T}{3}}$$



Here  $s_{31}\phi_1(t) = 0$   $S_{31} = 0$

$$\text{and } S_{32}\phi_2(t) = \begin{cases} \sqrt{\frac{T}{3}} \times \sqrt{\frac{3}{T}} & \text{for } T/3 \leq t \leq \frac{2T}{3} \\ 0 & \text{otherwise} \end{cases}$$

$$S_{32}\phi_2(t) = \begin{cases} 1 & \text{for } \frac{T}{3} \leq t \leq \frac{2T}{3} \\ 0 & \text{otherwise} \end{cases}$$

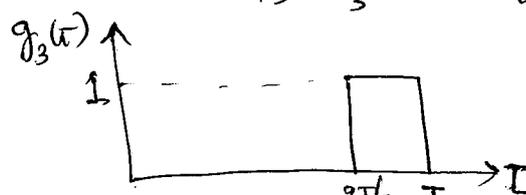
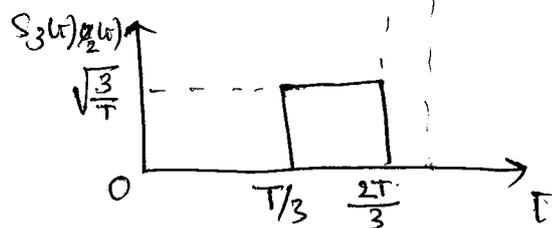
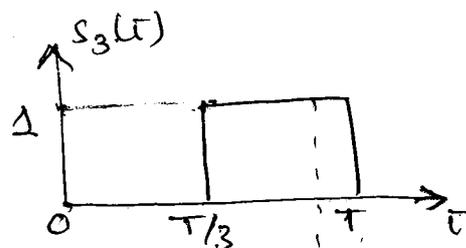
w.k.t

$$g_3(t) = s_3(t) - s_{31}\phi_1(t) - s_{32}\phi_2(t)$$

$$g_3(t) = s_3(t) - s_{31}\phi_1(t) - s_{32}\phi_2(t)$$

$$g_3(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq \frac{2T}{3} \\ 1 & \text{for } \frac{2T}{3} \leq t \leq T \\ 0 & \text{for } t \geq T \end{cases}$$

$$g_3(t) = \begin{cases} 1 & \text{for } \frac{2T}{3} \leq t \leq T \\ 0 & \text{elsewhere.} \end{cases}$$



$$Eg_3 = \int_0^T g_3^2(t) dt$$

$$= \int_{T/3}^{2T/3} (1)^2 dt$$

$$Eg_3 = \left[ t \right]_{T/3}^{2T/3}$$

$$Eg_3 = \frac{2T}{3} - \frac{T}{3}$$

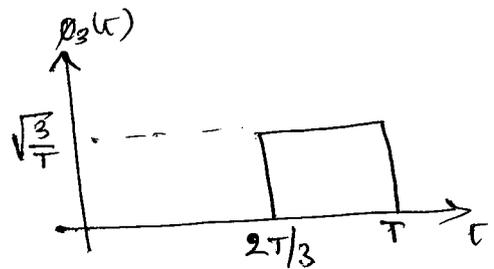
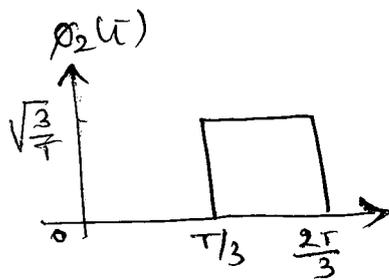
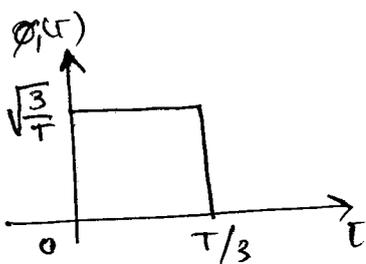
$$Eg_3 = \frac{T}{3}$$

$$\phi_3(t) = \frac{g_3(t)}{\sqrt{Eg_3}}$$

$$\phi_3(t) = \begin{cases} \frac{1}{\sqrt{\frac{T}{3}}} & \text{for } \frac{2T}{3} \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

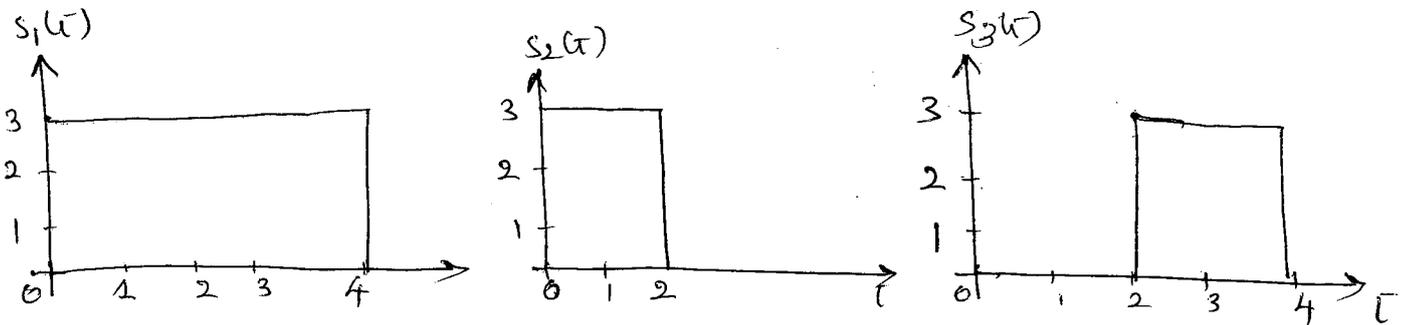
$$\phi_3(t) = \begin{cases} \sqrt{\frac{3}{T}} & \text{for } \frac{2T}{3} \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

Figure below shows orthonormal basis functions



### Problem ③

Three signals  $s_1(t)$ ,  $s_2(t)$  and  $s_3(t)$  are shown in Fig. Apply Gram-Schmidt procedure to obtain an orthonormal basis for the signals. Express signals  $s_1(t)$ ,  $s_2(t)$  and  $s_3(t)$  in terms of orthonormal basis function.



Soln: i) To obtain orthonormal basis function

$$\text{Here } s_3(t) = s_1(t) - s_2(t)$$

Hence we will obtain basis functions for  $s_1(t)$  and  $s_2(t)$  only.

To obtain  $\phi_1(t)$

Energy of  $s_1(t)$

$$E_1 = \int_0^T s_1^2(t) dt$$

$$E_1 = \int_0^4 3^2 dt = \int_0^4 9 dt = 9[4-0] = 36$$

$$E_1 = 36.$$

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{3}{\sqrt{36}} = \frac{3}{6} = \frac{1}{2} = \begin{cases} \frac{1}{2} & \text{for } 0 \leq t \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

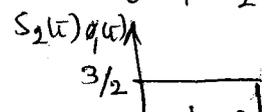
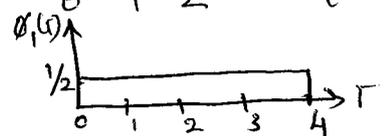
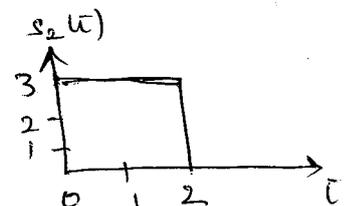
To obtain  $\phi_2(t)$

$$s_{2,1}(t) = \int_0^T s_2(t) \phi_1(t) dt = \int_0^2 3 \times \frac{1}{2} dt$$

$$= \frac{3}{2} [2-0]$$

$$s_{2,1}(t) = 3$$

$$s_{2,1}(t) = \begin{cases} 3 & 0 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$



$$g_2(t) = S_2(t) - S_{21}\phi_1(t)$$

$$S_{21}\phi_1(t) = \begin{cases} 3/2 & 0 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$g_2(t) = \begin{cases} 3/2 & 0 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$E_{g_2} = \int_0^T g_2^2(t) dt$$

$$E_{g_2} = \int_0^2 (3/2)^2 dt$$

$$E_{g_2} = \frac{9}{4} \int_0^2 dt$$

$$E_{g_2} = \frac{9}{4} [2 - 0]$$

$$E_{g_2} = \begin{cases} \frac{9}{2} & 0 \leq t \leq 2 \end{cases}$$

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{E_{g_2}}}$$

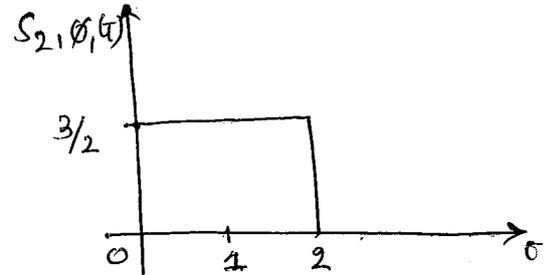
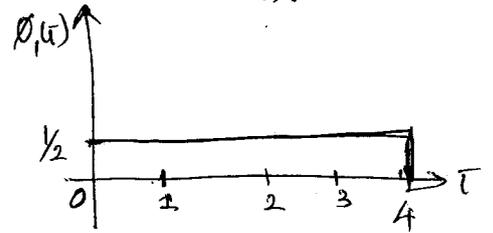
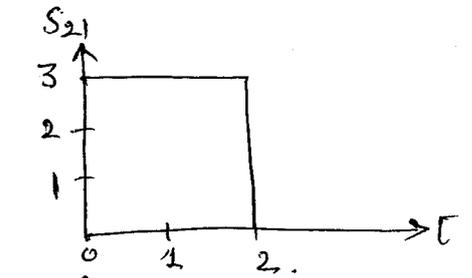
$$\phi_2(t) = \frac{3/2}{\sqrt{9/2}}$$

$$= \frac{3/2}{\frac{3}{\sqrt{2}}}$$

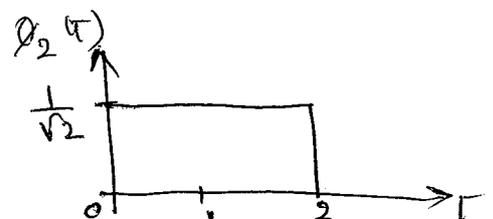
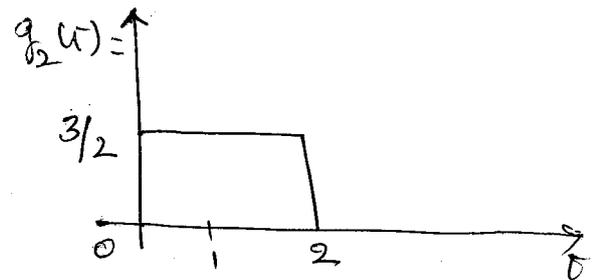
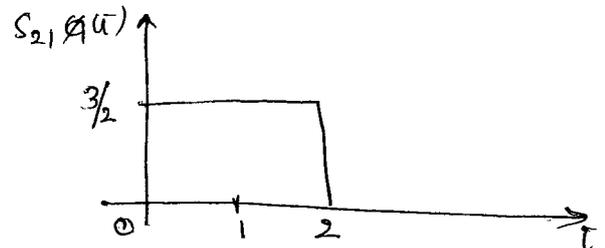
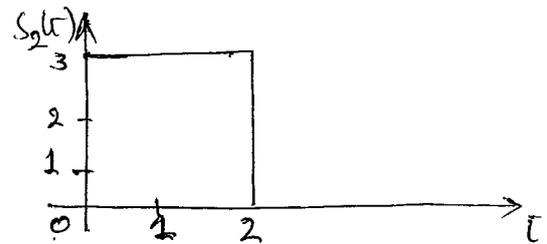
$$= 1/2 \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$\phi_2(t) = \frac{\sqrt{2}}{\sqrt{2} \times \sqrt{2}}$$

$$\phi_2(t) = \begin{cases} \frac{1}{\sqrt{2}} & 0 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$



$$g_2(t) = S_2(t) - S_{21}\phi_1(t)$$



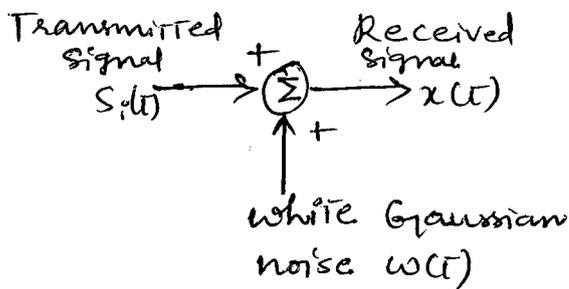
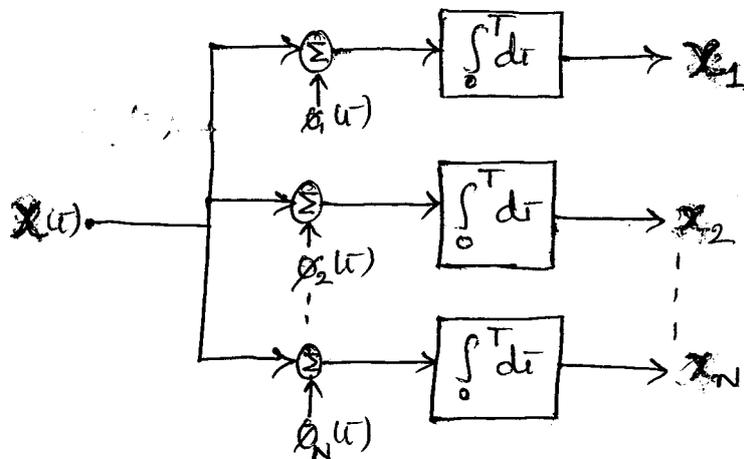
## Conversion of The Continuous AWGN Channel into a Vector Channel

- Let us consider that the noisy received signal is expressed by an random process  $x(t)$  which is given as

$$x(t) = s_i(t) + w(t), \quad \begin{cases} 0 \leq t \leq T \\ i = 1, 2, \dots, M. \end{cases} \quad - (1)$$

where  $w(t)$  is a sample function of the white Gaussian noise process  $W(t)$ . of zero mean and power spectral density  $\frac{N_0}{2}$ .

-  $s_i(t)$  is the transmitted signal. The received signal is applied to a bank of 'N' product integrators and correlators as shown in figure.



- The output of each correlator is a random variable which is given as  $x_j$ . We find that the output of correlator 'j' is the sample value of a random variable  $x_j$  whose sample value is defined by

$$x_j = \int_0^T x(t) \phi_j(t) dt \quad - (2)$$

Substitute eq (1) in eq (2)

$$x_j = \int_0^T [s_i(t) + w(t)] \phi_j(t) dt$$

$$x_j = \int_0^T s_i(t) \phi_j(t) dt + \int_0^T w(t) \phi_j(t) dt$$

w.k.t  $s_{ij} = \int_0^T s_i(t) \phi_j(t) dt$  and  $w_j = \int_0^T w(t) \phi_j(t) dt$

$$x_j = s_{ij} + w_j \quad \text{--- (3)} \quad j = 1, 2, \dots, N$$

From Eq (3)  $s_{ij}$  is the deterministic component of  $x_j$  due to the transmitted signal  $s_i(t)$  and  $w_j$  is the sample value of a random variable  $W_j$  due to the channel noise  $w(t)$ .

Let the new random variable  $x'(t)$  whose sample function  $x'(t)$  is related to the received signal  $x(t)$  is defined as.

$$x'(t) = x(t) - \sum_{j=1}^N x_j \phi_j(t) \quad \text{--- (4)}$$

Substitute Eq (1) and (2) in Eq (4)

$$x'(t) = s_i(t) + w(t) - \left\{ \sum_{j=1}^N (s_{ij} + w_j) \phi_j(t) \right\}$$

$$x'(t) = s_i(t) + w(t) - \sum_{j=1}^N s_{ij} \phi_j(t) - \sum_{j=1}^N w_j \phi_j(t)$$

w.k.t  $s_i(t) = \sum_{j=1}^N s_{ij} \phi_j(t)$

$$x'(t) = s_i(t) + w(t) - s_i(t) - \sum_{j=1}^N w_j \phi_j(t)$$

$$x'(t) = w(t) - \sum_{j=1}^N w_j \phi_j(t) \quad \text{--- (5)}$$

Since  $x'(t) = x(t) - \sum_{j=1}^N x_j \phi_j(t)$  therefore Equation (5)

can be written as

$$\boxed{x'(t) = w'(t)} \quad \text{--- (6)}$$

- From Equation (6) it's clear that the sample function  $x'(t)$  depends on the channel noise  $w'(t)$
- on the basis of Eq (4) and Eq (5) we may express the received signal as

$$x(t) = \sum_{j=1}^N x_j \phi_j(t) + x'(t)$$

$$x(t) = \sum_{j=1}^N x_j \phi_j(t) + w'(t) \quad \text{--- (7) since } x'(t) = w'(t)$$

- Here  $w'(t)$  is totally due to noise  $w(t)$  at the input of correlators.
- The correlator outputs  $x_1, x_2, \dots, x_N$  are Gaussian random variables. They are characterised completely by mean and variance.
- The received signal of Equation (7) can be represented by the vector form as follows.

$$x(t) = [x_1, x_2, \dots, x_N] \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ \vdots \\ \phi_N(t) \end{bmatrix} + w'(t) \quad \text{--- (8)}$$

- The AWGN channel can be thus represented by vector Equation as given by Eq (8)

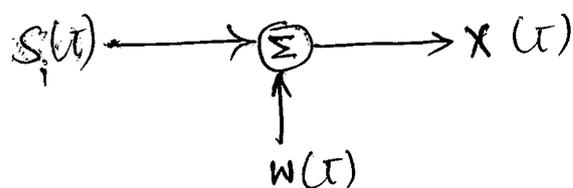
## Statistical characterization of the correlator outputs (8)

The random process  $x(t) = s_i(t) + w(t)$  is received by a bank of correlators. The output of the correlator is  $x_j = s_{ij} + w_j$ ,  $j = 1, 2, \dots, N$ . The noise  $w(t)$  is white Gaussian noise of zero mean & power spectral density of  $\frac{N_0}{2}$

- i) Determine mean value of  $x_j$
- ii) Determine variance of  $x_j$
- iii) show that  $x_j$  are mutually uncorrelated.

Soln - Here we are developing a statistical characterization of the set of 'N' correlator outputs.

- Let  $x(t)$  denote the stochastic (random) process, a sample function of which is represented by the received signal  $x(t)$ .
- Correspondingly  $x_j$  denotes the random variable whose sample value is represented by the correlator output  $x_j$ ,  $j = 1, 2, \dots, N$ .
- According to the AWGN channel model shown in below figure, the stochastic (random) process  $x(t)$  is a Gaussian process. It follows, that  $x_j$  is a Gaussian random variable for all 'j' and hence  $x_j$  is completely characterized by its mean & variance.



$$s_i(t) = \{s_1(t), s_2(t), \dots, s_m(t)\}$$

$$x_j(t) = \{x_1(t), x_2(t), \dots, x_N(t)\}$$

$$w_j(t) = \{w_1(t), w_2(t), \dots, w_N(t)\}$$

- From the figure  $W_j$  denote the random variable represented by the sample value  $w_j$  produced by the  $j$ th correlator in response to the white Gaussian noise component  $w(t)$ .

- The random variable  $W_j$  has zero mean because the channel noise process  $w(t)$  represented by  $w(t)$  in the above figure AWGN model, has zero mean by definition.

- But the mean of  $X_j$  depends only on  $s_{ij}$  therefore mean of  $X_j$  can be determined as follows

i) To obtain mean value of  $X_j$

The mean value of  $X_j$  is given as

$$\mu_{X_j} = E[X_j] \quad - (1)$$

where  $E[X_j]$  is the expected value of  $X_j$

$$\text{We know that } X_j = s_{ij} + W_j \quad - (2)$$

Substitute eq (2) in eq (1)

$$\mu_{X_j} = E[s_{ij} + W_j]$$

$$\mu_{X_j} = E[s_{ij}] + E[W_j] \quad - (3)$$

We know that mean value of white Gaussian noise is zero. Hence  $E[W_j]$  in eq (3) becomes zero i.e.  $E[W_j] = 0$

Therefore the mean value of  $X_j$  will be

$$\boxed{\mu_{X_j} = s_{ij}} \quad - (4)$$

ii) To obtain variance of  $X_j$

- The definition of variance is given by the equation

$$\sigma_{X_j}^2 = \text{Var}[X_j] \quad - (5)$$

Note: - To find variance of  $X_j$

step 1: Find  $\mu$  of  $X_j$

step 2:  $(X_j - \mu)$

step 3:  $(X_j - \mu)^2$

$$\sigma_{X_j}^2 = E \left[ (X_j - \mu_{X_j})^2 \right]$$

w.k.T  $\mu_{X_j} = S_{ij}$

$$\sigma_{X_j}^2 = E \left[ (X_j - S_{ij})^2 \right]$$

w.k.T  $X_j = S_{ij} + W_j$

$$\sigma_{X_j}^2 = E \left[ (S_{ij} + W_j - S_{ij})^2 \right]$$

$$\sigma_{X_j}^2 = E \left[ W_j^2 \right] \quad \text{--- (6)}$$

We know that  $W_j = \int_0^T w(t) \phi_j(t) dt$  --- (7)

Substitute eq (7) in Equation (6) and we may expand this as follows.

$$\sigma_{X_j}^2 = E \left[ \int_0^T w(t) \phi_j(t) dt \int_0^T w(u) \phi_j(u) du \right] \quad \text{--- (8)}$$

Note: -  $E \left[ W_j^2 \right]$  is written as two separate integration terms with different indices of integration.

Equation (8) can be rewritten as

$$\sigma_{X_j}^2 = E \left[ \int_0^T \int_0^T \phi_j(t) \phi_j(u) w(t) w(u) dt du \right] \quad \text{--- (9)}$$

- Interchange the order of integration and expectation which can do because they are both linear operations we obtain.

$$\sigma_{X_j}^2 = \int_0^T \int_0^T \phi_j(t) \phi_j(u) E \left[ w(t) w(u) \right] dt du \quad \text{--- (10)}$$

Here  $E \left[ w(t) w(u) \right]$  represents autocorrelation function of the noise process  $w(t)$ .

$$E \left[ w(t) w(u) \right] = R_w(t, u) \quad \text{--- (11)}$$

Substitute eq (11) in eq (10)

$$\sigma_{x_j}^2 = \int_0^T \int_0^T \phi_i(t) \phi_j(u) R_W(t, u) dt du \quad - (11)$$

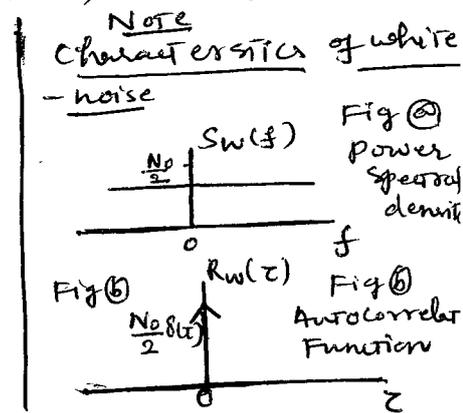
Since the noise is stationary,  $R_W(t, u)$  depends only on the time difference  $(t-u)$ . Furthermore  $W(t)$  is white with constant power spectral density  $\frac{N_0}{2}$  and we may express  $R_W(t, u)$  as.

$$R_W(t, u) = \left(\frac{N_0}{2}\right) \delta(t-u) \quad - (12)$$

Therefore substitute eq (12) in eq (11) and using the shifting property of the delta function  $\delta(t)$  we get

$$\sigma_{x_j}^2 = \frac{N_0}{2} \int_0^T \int_0^T \phi_i(t) \phi_j(u) \delta(t-u) dt du$$

$$\sigma_{x_j}^2 = \frac{N_0}{2} \int_0^T \phi_j^2(t) dt \quad - (13)$$



Since the above equation has non-zero value only at  $t=u$ .

→ The integration term in the above equation (13) represents  $\phi_j(t)$  which is always equal to unity. Hence the expression for noise variance  $\sigma_{x_j}^2$  reduces to

$$\sigma_{x_j}^2 = \frac{N_0}{2} \quad \text{for all } j \quad - (14)$$

→ Thus from the above equation it's clear that all the correlator outputs denoted by  $x_j$  with  $j=1, 2, \dots, N$  have a variance equal to the power spectral density  $\frac{N_0}{2}$  of the noise process  $W(t)$ . (15) The variance of correlator output is equal to power spectral density

iii) To show that  $X_j$  are mutually uncorrelated

— Since the basis function  $\phi_j(t)$  form an orthonormal set, and  $X_j$  and  $X_k$  are mutually uncorrelated as

$$\text{Cov}[X_j, X_k] = E[(X_j - \mu_{X_j})(X_k - \mu_{X_k})]$$

$$\text{w.k.t } \mu_{X_j} = S_{ij}$$

$$\mu_{X_k} = S_{ik}$$

$$= E[(X_j - S_{ij})(X_k - S_{ik})]$$

Here  $X_j = S_{ij} + W_j$

$W_j = X_j - S_{ij}$      by      $W_k = X_k - S_{ik}$

$$\text{Cov}[X_j, X_k] = E[W_j W_k]$$

$$= E\left[\int_0^T W(t) \phi_j(t) dt \int_0^T W(u) \phi_k(u) du\right]$$

$$\text{Cov}[X_j, X_k] = \int_0^T \int_0^T \phi_j(t) \phi_k(u) E[W(t)W(u)] dt du$$

Here  $E[W(t), W(u)] = R_W(t, u)$

$$\text{Cov}[X_j, X_k] = \int_0^T \int_0^T \phi_j(t) \phi_k(u) R_W(t, u) dt du$$

w.k.t  $R_W(t, u) = \left(\frac{N_0}{2}\right) \delta(t-u)$

$$\text{Cov}[X_j, X_k] = \int_0^T \int_0^T \phi_j(t) \phi_k(u) \left(\frac{N_0}{2}\right) \delta(t-u) dt du$$

$$= \frac{N_0}{2} \int_0^T \int_0^T \phi_j(t) \phi_k(u) \delta(t-u) dt du$$

Since the above equation is non-zero for  $t=u$  because  $\delta(t-u)$

$$= \frac{N_0}{2} \int_0^T \int_0^T \phi_j(t) \phi_k(t) dt du = \frac{N_0}{2} \times 0,$$

Since  $\phi_j$  &  $\phi_k$  are orthogonal to each other i.e.  $\int_0^T \phi_j(t) \phi_k(t) dt = 0$

$$\boxed{\text{Cov}[X_j, X_k] = 0, \quad j \neq k}$$

- In the above equation  $x_1, x_2, \dots, x_N$  are Gaussian random variables, and they are statistically independent. The vector of 'N' random variables can be written as

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

- The elements in the above equation are independent Gaussian random variables with mean value is equal to  $S_{ij}$  and variance equal to  $\frac{N_0}{2}$ .

### Optimum Receivers Using Coherent Detection

- In digital communication system there are three optimum receiver techniques used for detection of message signal at the receiver. They are

- ① Maximum Likelihood Decoding
- ② Correlation Receiver
- ③ Matched Filter Receiver.

- The transmitted signal  $s_i(t)$  is often corrupted by noise when it is received. Such a received signal is represented by

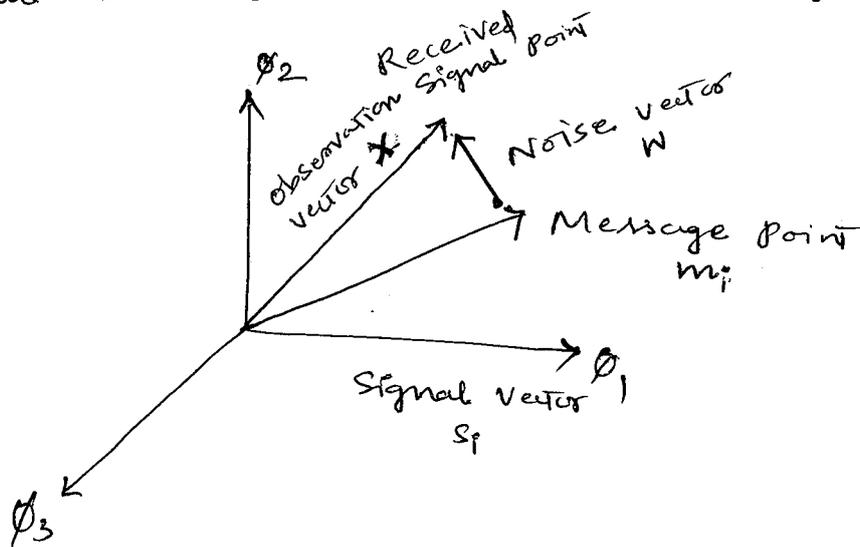
$$r(t) = s_i(t) + w(t) \quad \text{for } 0 \leq t \leq T \quad \text{--- ①}$$

$$i = 1, 2, \dots, M.$$

Here  $w(t)$  is the sample function of additive white Gaussian noise process  $W(t)$ .

- The main job of the receiver is to observe the signal  $x(t)$  and make best estimate of  $s_i(t)$ .

- The transmitted signal vector  $s_i$ , the observation vector 'X' and the noise vector 'W' can be represented in the N-dimensional Euclidean space is called a signal constellation as shown in fig.



- From the above constellation diagram shows that for  $N=3$  (i.e. 3-dimensions), observe that the received signal point shifts from message point ( $m_i$ ) by a noise vector 'W'.

- The detector observes the vector 'X' and performs the mapping to an estimate  $\hat{m}$  of the transmitted message point  $m_i$ . This mapping is done in such a way that the average probability of symbol error is minimum in the decision.

### Maximum Likelihood (ML) Decoding

- Maximum likelihood, also called the maximum likelihood method, is the procedure of finding the value of one or more parameters for a given statistic which makes the known likelihood distribution a maximum.

- The maximum likelihood decision rule is simply choose the message point closest to the received signal point which is satisfy

- Let the observation vector be 'x'. The decision made as  $\hat{m} = m_i$ . The average probability of symbol error in this decision, which denoted by

$$P_e(m_i|x) = 1 - P(m_i|sent|x)$$

where  $P_e(m_i|x)$  indicates average probability of symbol error when 'x' is the observation vector and message  $m_i$  is selected.

- To minimize the error probability  $P_e(m_i|x)$  given by above equation the optimum decision rule can be stated as.

$$\text{Set } \hat{m} = m_i \text{ if } \left\{ \begin{array}{l} \text{Note: The decision rule described} \\ \text{in below equation is the maximum} \\ \text{a posterior probability (MAP) rule.} \\ \text{Correspondingly rule is called MAP detector.} \end{array} \right.$$

$$P(m_i|sent|x) \geq P(m_k|sent|x) \text{ for all } k \neq i$$

and  $k=1, 2, \dots, M$

- This decision rule can be represented graphically. Let 'Z' denote the N-dimensional space of all possible vector 'x'. And this region be partitioned into 'M' decision regions  $Z_1, Z_2, \dots, Z_M$ . The decision rule can then be written as

- vector x lies in region  $R_i$  if  $\ln[f_x(x/m_k)]$  is maximum for  $k=i$ .

- Here  $f_x(x/m_k)$  is the likelihood function which results when symbol  $m_k$  is transmitted. This rule is called maximum likelihood and the corresponding detector which uses this rule is called maximum likelihood detector.

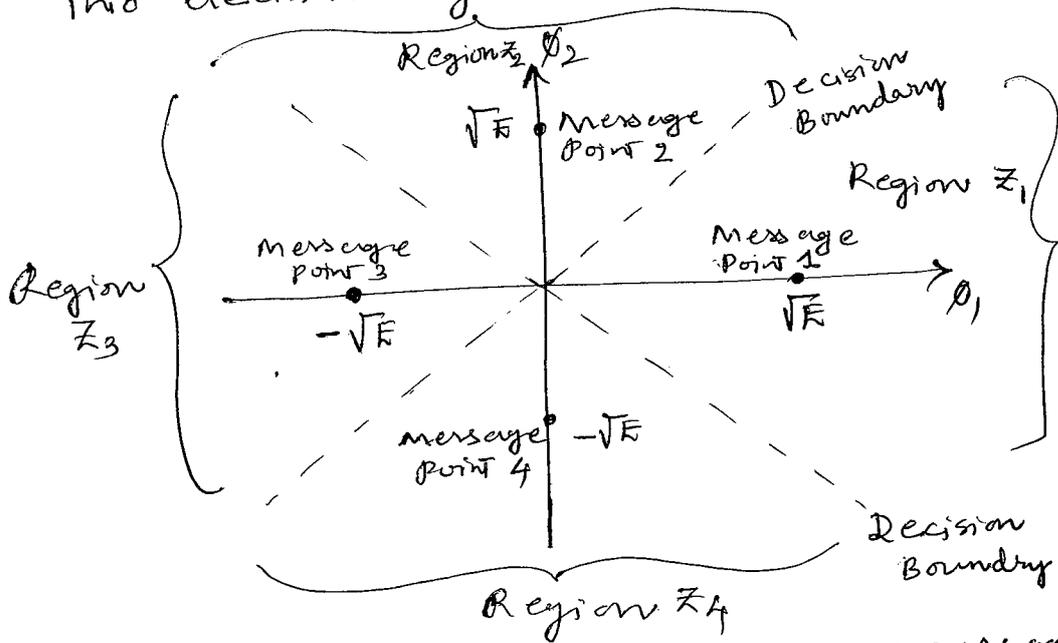
- The decision rule of equation  $\ln[f_x(x/m_k)]$  can be further written alternately as

vector x lies in region  $R_i$  if  $\|x - s_k\|$  is minimum for  $k=i$

- Here  $\|x - s_k\|$  is the distance between received signal point and the message point. The maximum likelihood decision rule chooses the message point closest to the received signal point.

### An Example of maximum likelihood decision for $N=2$ and $M=4$

- Figure illustrates the partitioning of the observation space into decision regions for the case when  $N=2$  and  $M=4$



- From the above figure, the message signals are transmitted with equal energy ' $E$ ' and equal probability.

- According to maximum likelihood decision rule, a maximum likelihood decoder computes the log-likelihood functions as metrics for all the ' $M$ ' possible message symbols, compares them and then decides in favour of the maximum.

# Example for Maximum a posteriori (MAP) and Maximum likelihood (ML) decoding

Assume two messages

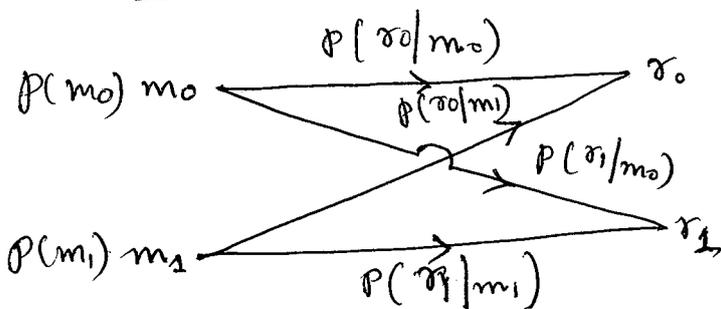
0  $\rightarrow$   $m_0$  message  
 1  $\rightarrow$   $m_1$  message.

When  $m_0$  is intended for  $r_0$  (received 0),  
 $m_1$  is intended for  $r_1$  (received 1) } no noise

Assume If noise is present in the channel  
 Transmission probability is defined as.

$P(r_0|m_0)$  = probability that  $r_0$  is received when  $m_0$  is TXd  
 $P(r_0|m_1)$  = probability that  $r_0$  is received when  $m_1$  is TXd.  
 $P(r_1|m_0)$  = probability that  $r_1$  is received when  $m_0$  is TXd  
 $P(r_1|m_1)$  = probability that  $r_1$  is received when  $m_1$  is TXd

## channel model



Here  $P(m_0)$  and  $P(m_1)$  are called prior probabilities

To Develop Algorithm for maximum likelihood by using MAP  
 (a) maximum a posteriori probabilities

$P(m_0|r_0)$  = probability that  $m_0$  is Transmitted given  $r_0$  is received  
 $P(m_0|r_1)$  = probability that  $m_0$  is Transmitted given  $r_1$  is received  
 $P(m_1|r_0)$  = probability that  $m_1$  is Transmitted given  $r_0$  is received  
 $P(m_1|r_1)$  = probability that  $m_1$  is Transmitted given  $r_1$  is received

Assume If  $r_0$  is received. (Algorithm Rule)

$P(m_0|r_0) > P(m_1|r_0)$  —  $m_0$  is message (correct)  
 $P(m_0|r_0) < P(m_1|r_0)$  —  $m_1$  is message (Incorrect)

Assume If  $r_1$  is received (Algorithm Rule)

$P(m_1|r_1) > P(m_0|r_1)$  —  $m_1$  is message (correct)  
 $P(m_1|r_1) < P(m_0|r_1)$  —  $m_0$  is message (Incorrect)

→ The maximum A posteriori (MAP) provides optimum receiver with minimum probability of errors.

Multiply  $p(r_0)$  to correct message of  $m_0$  &  $m_1$

$$p(r_0) p(m_0|r_0) > p(m_1|r_0) p(r_0)$$

W.K.T Conditional probability

$$p(A/B) = \frac{p(A, B)}{p(B)} \quad \& \quad p(B/A) = \frac{p(A, B)}{p(A)}$$

$p(A, B) = p(A/B)p(B)$  and  $p(B/A)p(A)$  — Joint probability

$$p(r_0|m_0) p(m_0) > p(r_0|m_1) p(m_1) \text{ — message } m_0$$

~~Imp~~  $p(r_1) p(m_1|r_1) > p(m_0|r_1) p(r_1)$

$$p(r_1|m_1) p(m_1) > p(r_1|m_0) p(m_0) \text{ — message } m_1$$

$$p(c) = p(r_0|m_0) p(m_0) + p(r_1|m_1) p(m_1)$$

$p(c)$  = correct probability  
Maximum likelihood decoding

$$p(m_0) = p(m_1) = \frac{1}{2} \cdot (\text{Equiprobable})$$

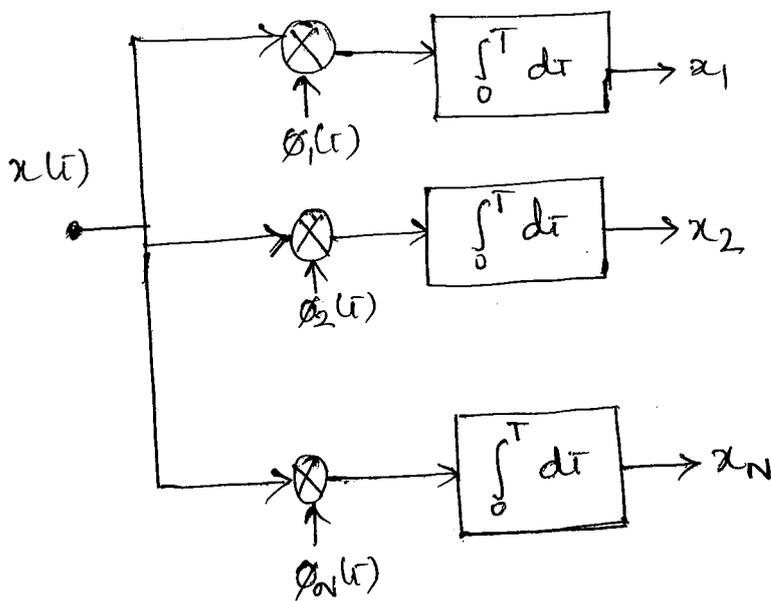
$$P_e = 1 - p(c)$$

## Correlation Receiver

- The Optimum Receiver for an AWGN channel and for the case when the transmitted signals  $s_1(t), s_2(t) \dots s_M(t)$  are equally likely is called a correlation receiver.
- The Correlation Receiver consists of two subsystems

① Detector as shown in below figure, which consists of  $M$  correlators supplied with a set of orthonormal basis functions  $\phi_1(t), \phi_2(t) \dots \phi_N(t)$  that are generated locally. This bank of correlators operates on the received signal  $x(t), 0 \leq t \leq T$  to produce the observation vector  $X$ .

Figure: Detector @ Demodulator.

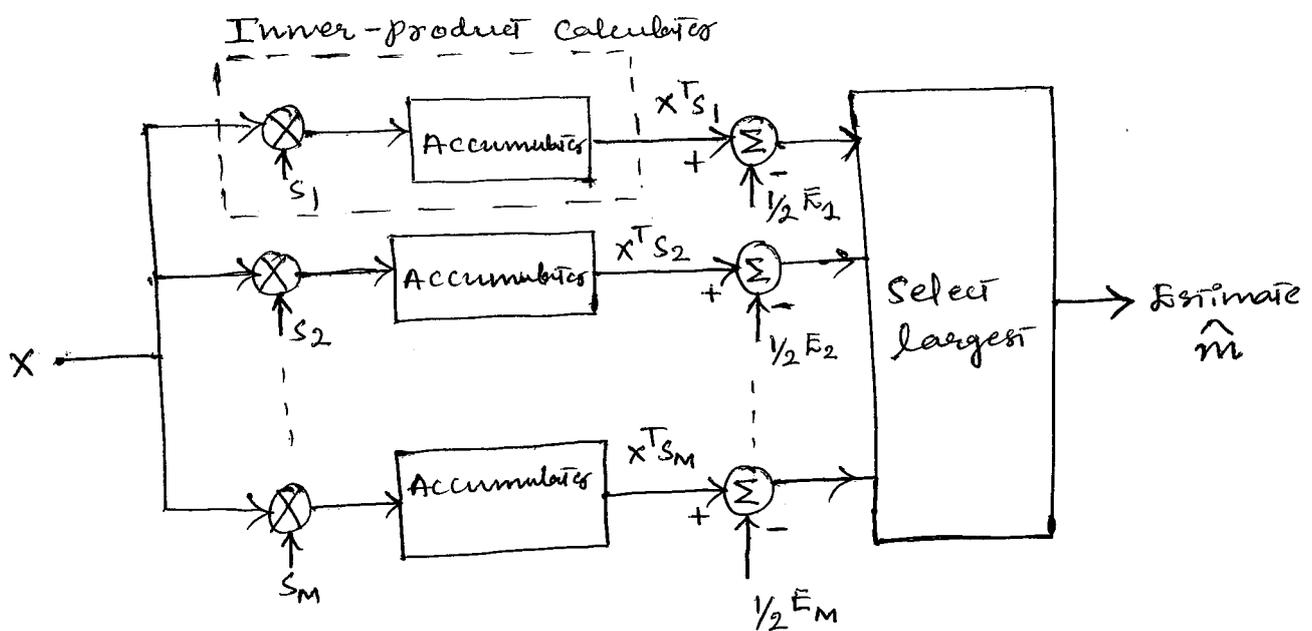


② Maximum-likelihood decoder shown in figure below, which operates on the observation vector  $X$  to produce an estimate  $\hat{m}$  of the transmitted symbol  $m_i, i = 1, 2, \dots, M$  in such way that the average probability of symbol error is minimized.

- In accordance with the maximum likelihood decision rule the decoder multiplies the  $N$  elements of the observation vector  $X$  by the corresponding ' $N$ ' elements of each of  $M$  signal vectors  $s_1, s_2 \dots s_M$ .

- Then, the resulting products are successively summed in accumulators to form the corresponding set of inner products  $\{x^T s_k \mid k = 1, 2, \dots, M\}$ .
- Next, the inner products are corrected for the fact that the transmitted signal energies may be unequal.
- Finally, the largest one in the resulting set of numbers is selected and an appropriate decision on the transmitted message is thereby made.

Figure: Signal Transmission decoder.



- It is called correlator since it correlates the received signal vector  $x$  with a stored replica of the known signal  $s_i(t)$ .

## Matched Filters

- The matched filter is the optimal linear filter for maximizing the signal-to-noise ratio (SNR) in the presence of additive stochastic (Random) noise.
- Matched filters is obtained by correlating a known signal @ template with an unknown signal to detect the presence of the template in the unknown signal.
- This is equivalent to convolving the unknown signal with a conjugated time reversed version of the template.

## Requirements of detection receiver

- 1) Signal to noise ratio of the receiver must be improved.
- 2) The signal must be checked at the instant in bit period, when signal to noise ratio is maximum.
- 3) The error probability should be minimum.

The above all the requirements fulfilled by the matched filters.

ie \*

- \* It satisfies all three requirements

- \* It is called matched filter since its impulse response is matched to the shape of the input signal.

- The maximum signal component occurs at  $t = T$  (ie sampling instant) and has magnitude  $E$ , ie energy of the signal  $x(t)$ .

## Matched Filter Receiver

- Consider a linear time-invariant filter with impulse response  $h_j(t)$ .
- The received signal is  $x(t)$  operating as input the resulting filter output is defined by the convolution integral.

$$y_j(t) = \int_{-\infty}^{\infty} x(z) h_j(t-z) dz \quad - (1)$$

- To proceed further to evaluate the above integral over the transmitted symbol duration between 0 to T. i.e.  $0 \leq t \leq T$ , and replace  $z$  with  $\tau$ .

$$y_j(t) = \int_0^T x(\tau) h_j(T-\tau) d\tau \quad - (2)$$

- Consider a detector based on the bank of correlator. The output of the  $j$ th correlator is defined by

$$x_j = \int_0^T x(t) \phi_j(t) dt \quad - (3)$$

- From eq (2) and (3) we can make  $y_j(t)$  is equal to  $x_j$ .

$$h_j(T-\tau) = \phi_j(\tau) \quad \text{for } 0 \leq \tau \leq T, \text{ and } j=1, 2, \dots, M$$

Equivalently we may express the impulse response of the filter as

$$h_j(\tau) = \phi_j(T-\tau) \quad \text{for } 0 \leq \tau \leq T, \text{ and } j=1, 2, \dots, M \quad - (4)$$

- The generalised condition can be described for the above equation as

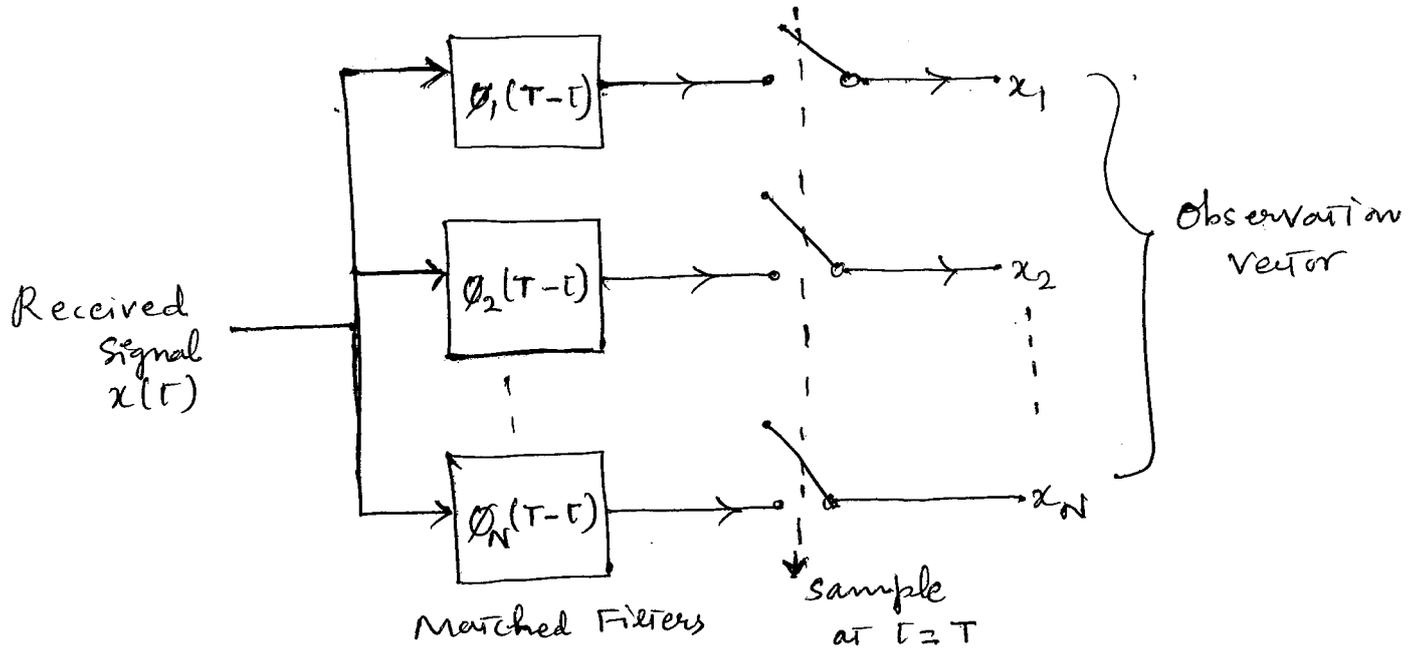
"Given a pulse  $\phi(t)$  occupying the interval  $0 \leq t \leq T$ , a linear time-invariant filter is said to be matched to the signal  $\phi(t)$  if its impulse response  $h(t)$  satisfies the condition"

$$h(t) = \phi(T-t) \quad \text{for } 0 \leq t \leq T$$

- A time-invariant filter defined in above statement is called as matched filter.

- An optimum receiver using matched filters in place of correlators is called a matched-filter receiver.

- Figure shows detector part of matched filter receiver.



# Orthonormal

- In linear algebra, two vectors in an inner product space are orthonormal if they are orthogonal & unit vectors.
- A set of vectors form an orthonormal set if all vectors in the set are mutually orthogonal and all of unit length.
- An orthonormal set which forms a basis is called an orthonormal basis

Ex Let  $V$  be an inner product space.

A set of vectors

$\{u_1, u_2, \dots, u_n, \dots\} \in V$  is called orthonormal if and only if  $\forall i, j: \langle u_i, u_j \rangle = \delta_{ij}$

Where  $\delta_{ij}$  is the Kronecker delta and  $\langle \cdot, \cdot \rangle$  is the inner product

(Element)  $\in \Rightarrow$  set membership

(Set)  $\{ \} \Rightarrow$  set (a collection of elements)

$A \cap B \rightarrow \cap$  (Intersection)

Objects that belong to set A & set B

$A \cup B \rightarrow$  Union (Objects that belong to set A  $\cup$  set B)

The inner product becomes a dot product of vectors

$\forall$  for all,  
| such that

## Basic Formula

If  $B$  is an orthogonal basis of  $H$  then every element  $x$  of  $H$  may be written as

$$x = \sum_{b \in B} \frac{\langle x, b \rangle}{\|b\|^2} b$$

$\prod$  - Capital  $\prod$  Product of all values  
in range of series

$P(A|B)$  - conditional probability

of probability of event 'A' given  
event 'B' occurred.

$|A|$  - cardinality - The number of  
elements of set 'A'

AUGIN

Additive - Because it is added to any noise that  
might be intrinsic to the information system.

White - Refers to the idea that it has uniform  
power across the frequency band for information  
system. (white which has uniform emission  
at all frequencies in the visible spectrum)

Gaussian - It has a normal distribution in the  
time domain with an average time  
domain value of zero.

Summation of many random process will tend to  
have distribution called Gaussian @ Normal

Eg: - Thermal noise, shot noise, black body  
radiation from earth, and other warm objects.  
and from celestial sources such as sun.

Gram-Schmidt (Orthogonalization)

Gram-Schmidt orthogonalization also called the Gram  
Schmidt process, is a procedure which takes a non-orthogonal  
set of linearly independent functions and constructs an  
orthogonal basis over an arbitrary interval with  
respect to an arbitrary weighting function  $w(x)$ .

## Orthogonal sets

- Let  $V$  be a vector space with an inner product.

Definition:- Non-zero vectors  $v_1, v_2, \dots, v_k \in V$  form an orthogonal set if they are orthogonal to each other  $\langle v_i, v_j \rangle = 0$  for  $i \neq j$

If, in addition, all vectors are of unit norm  $\|v_i\| = 1$ , then  $v_1, v_2, \dots, v_k$  is called an orthonormal set.

Theorem:- Any orthogonal set is linearly independent

## The Gram-Schmidt Orthogonalization process

- Let  $V$  be a vector space with an inner product. Suppose  $x_1, x_2, \dots, x_n$  is a basis for  $V$ . Let

$$v_1 = x_1$$

$$v_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$v_3 = x_3 - \frac{\langle x_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle x_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

### Normalization

Let  $V$  be a vector space with an inner product. Suppose  $v_1, v_2, \dots, v_n$  is an orthogonal basis for  $V$

$$\text{let } w_1 = \frac{v_1}{\|v_1\|}, w_2 = \frac{v_2}{\|v_2\|}, \dots, w_n = \frac{v_n}{\|v_n\|}$$

Then  $w_1, w_2, \dots, w_n$  is an orthonormal basis of  $V$

### Orthogonalization

Suppose  $x_1, x_2, \dots, x_n$  is a basis for an inner product space  $V$ . Let

$$v_1 = x_1, w_1 = \frac{v_1}{\|v_1\|}$$

$$v_2 = x_2 - \langle x_2, w_1 \rangle w_1, w_2 = \frac{v_2}{\|v_2\|}$$

$$v_3 = x_3 - \langle x_3, w_1 \rangle w_1 - \langle x_3, w_2 \rangle w_2, w_3 = \frac{v_3}{\|v_3\|}$$

Definition : Let  $u \in V$  be no vectors. The projection of the vectors  $v$  on  $u$  is defined as follows

$$B = \{u_1, u_2, \dots, u_n\}$$

$$\text{Let } v_1 = u_1$$

$$v_2 = u_2 - \text{Proj}_{v_1} u_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

Note

If  $u_1, u_2, u_3$  is not orthogonal. Then  $v_1 \cdot v_2$  also not equal to 0. Then we use Gram-Schmidt procedure.



