# Module 4: Transportation Problem and Assignment problem

**Transportation problem** is a special kind of **Linear Programming Problem (LPP)** in which goods are transported from a set of sources to a set of destinations subject to the supply and demand of the sources and destination respectively such that the total cost of transportation is minimized. It is also sometimes called as Hitchcock problem.

**Types of Transportation problems:**

**Balanced:** When both supplies and demands are equal then the problem is said to be a balanced transportation problem.

**Unbalanced:** When the supply and demand are not equal then it is said to be an unbalanced transportation problem. In this type of problem, either a dummy row or a dummy column is added according to the requirement to make it a balanced problem. Then it can be solved similar to the balanced problem.

**Methods to Solve:**

To find the initial basic feasible solution there are three methods:

1. NorthWest Corner Cell Method.
2. Least Call Cell Method.
3. Vogel's Approximation Method (VAM).

**Basic structure of transportation problem:**

Destination

| Source | D1 | D2 | D3 | D4 | Supply($s_i$) |
|---|---|---|---|---|---|
| O1 | $C_{11}$ | $C_{12}$ | $C_{13}$ | $C_{14}$ | $s_1$ |
| O2 | $C_{21}$ | $C_{22}$ | $C_{23}$ | $C_{24}$ | $s_2$ |
| O3 | $C_{31}$ | $C_{32}$ | $C_{33}$ | $C_{34}$ | $s_3$ |
| O4 | $C_{41}$ | $C_{42}$ | $C_{43}$ | $C_{44}$ | $s_4$ |
| Demand ($d_j$): | $d_1$ | $d_2$ | $d_3$ | $d_4$ | |

In the above table **D1**, **D2**, **D3** and **D4** are the destinations where the products/goods are to be delivered from different sources **S1**, **S2**, **S3** and **S4**. $\mathbf{S_i}$ is the supply from the source $\mathbf{O_i}$. $\mathbf{d_j}$ is the demand of the destination $\mathbf{D_j}$. $\mathbf{C_{ij}}$ is the cost when the product is delivered from source $\mathbf{S_i}$ to destination $\mathbf{D_j}$.

**a) Transportation Problem : (NorthWest Corner Method)**

An introduction to Transportation problem has been discussed in the previous Section, in this, finding the initial basic feasible solution using the NorthWest Corner Cell Method will be discussed.

Destination

| Source | D1 | D2 | D3 | D4 | Supply |
|---|---|---|---|---|---|
| O1 | 3 | 1 | 7 | 4 | 300 |
| O2 | 2 | 6 | 5 | 9 | 400 |
| O3 | 8 | 3 | 3 | 2 | 500 |
| Demand: | 250 | 350 | 400 | 200 | 1200 |

**Explanation:** Given three sources **O1**, **O2** and **O3** and four destinations **D1**, **D2**, **D3** and **D4**. For the sources **O1**, **O2** and **O3**, the supply is **300**, **400** and **500** respectively.

The destinations **D1**, **D2**, **D3** and **D4** have demands **250**, **350**, **400** and **200** respectively.

**Solution:** According to North West Corner method, **(O1, D1)** has to be the starting point i.e. the north-west corner of the table. Each and every value in the cell is considered as the cost per transportation. Compare the demand for column **D1** and supply from the source **O1** and allocate the minimum of two to the cell **(O1, D1)** as shown in the figure.

The demand for Column **D1** is completed so the entire column **D1** will be canceled. The supply from the source **O1** remains **300 – 250 = 50**.

Destination

| Source | D1 | D2 | D3 | D4 | Supply |
|---|---|---|---|---|---|
| O1 | 250 3 | 1 | 7 | 4 | ~~300~~ 50 |
| O2 | 2 | 6 | 5 | 9 | 400 |
| O3 | 8 | 3 | 3 | 2 | 500 |
| Demand: | ~~250~~ 0 | 350 | 400 | 200 | 1200 |

Now from the remaining table i.e. excluding column **D1**, check the north-west corner i.e. **(O1, D2)** and allocate the minimum among the supply for the respective column and the rows. The supply from **O1** is **50** which is less than the demand for **D2** (i.e. 350), so allocate **50** to the cell **(O1, D2)**.

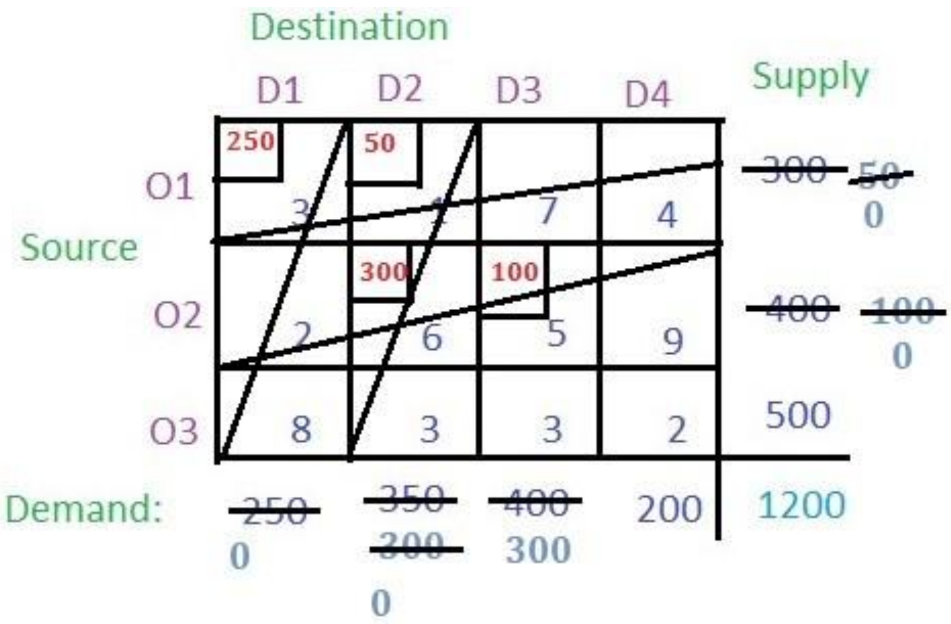
Since the supply from row **O1** is completed cancel the row **O1**. The demand for column **D2** remain **350 – 50 = 50**.

| Source \ Destination | D1 | D2 | D3 | D4 | Supply |
|---|---|---|---|---|---|
| O1 | 250 3 | 50 1 | 7 | 4 | ~~300~~ ~~50~~ 0 |
| O2 | 2 | 6 | 5 | 9 | 400 |
| O3 | 8 | 3 | 3 | 2 | 500 |
| Demand: | ~~250~~ 0 | ~~350~~ 300 | 400 | 200 | 1200 |

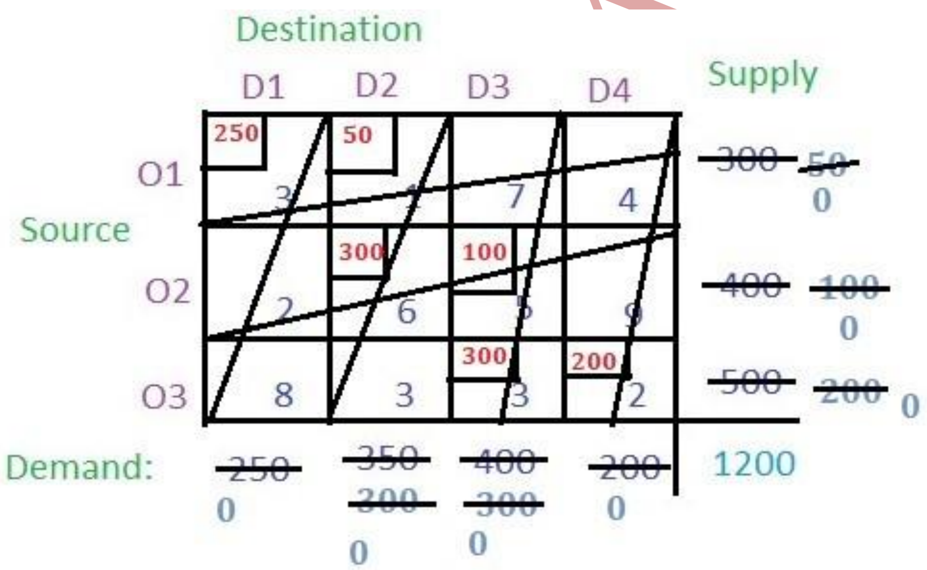
From the remaining table the north-west corner cell is **(O2, D2)**. The minimum among the supply from source **O2** (i.e 400) and demand for column **D2** (i.e 300) is **300**, so allocate **300** to the cell **(O2, D2)**. The demand for the column **D2** is completed so cancel the column and the remaining supply from source **O2** is **400 – 300 = 100**.

| Source \ Destination | D1 | D2 | D3 | D4 | Supply |
|---|---|---|---|---|---|
| O1 | 250 3 | 50 1 | 7 | 4 | ~~300~~ ~~50~~ 0 |
| O2 | 2 | 300 6 | 5 | 9 | ~~400~~ 100 |
| O3 | 8 | 3 | 3 | 2 | 500 |
| Demand: | ~~250~~ 0 | ~~350~~ ~~300~~ 0 | 400 | 200 | 1200 |

Now from remaining table find the north-west corner i.e. **(O2, D3)** and compare the **O2**supply (i.e. 100) and the demand for **D2** (i.e. 400) and allocate the smaller (i.e. 100) to the cell **(O2, D2)**. The supply from **O2** is completed so cancel the row **O2**. The remaining demand for column **D3** remains **400 – 100 = 300**.




Proceeding in the same way, the final values of the cells will be:




**Note:** In the last remaining cell the demand for the respective columns and rows are equal which was cell **(O3, D4)**. In this case, the supply from **O3** and the demand for **D4** was **200**which was allocated to this cell. At last, nothing remained for any row or column.

Now just multiply the allocated value with the respective cell value (i.e. the cost) and add all of them to get the basic solution i.e. **(250 * 3) + (50 * 1) + (300 * 6) + (100 * 5) + (300 * 3) + (200 * 2) = 4400**

### b) Transportation Problem: (Least Cost Cell Method)

The **North-West Corner** method has been discussed in the previous session. In this session, the **Least Cost Cell** method will be discussed.

Destination

| Source | D1 | D2 | D3 | D4 | Supply |
|---|---|---|---|---|---|
| O1 | 3 | 1 | 7 | 4 | 300 |
| O2 | 2 | 6 | 5 | 9 | 400 |
| O3 | 8 | 3 | 3 | 2 | 500 |
| Demand: | 250 | 350 | 400 | 200 | 1200 |

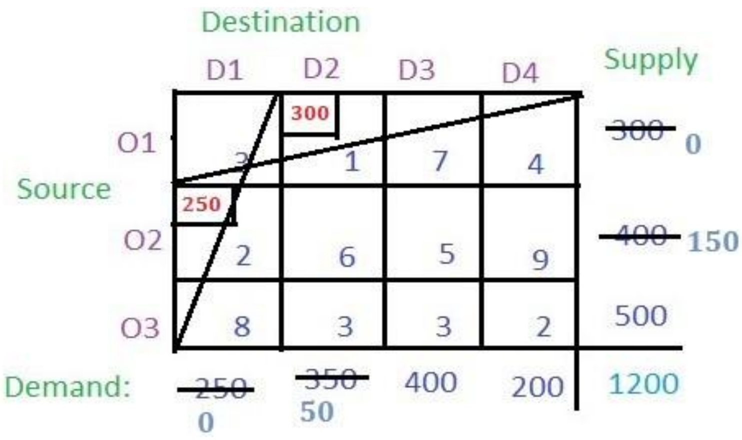
**Solution:** According to the Least Cost Cell method, the least cost among all the cells in the table has to be found which is **1** (i.e. cell **(O1, D2)**).

Now check the supply from the row **O1** and demand for column **D2** and allocate the smaller value to the cell. The smaller value is **300** so allocate this to the cell. The supply from **O1** is completed so cancel this row and the remaining demand for the column **D2** is **350 – 300 = 50**.

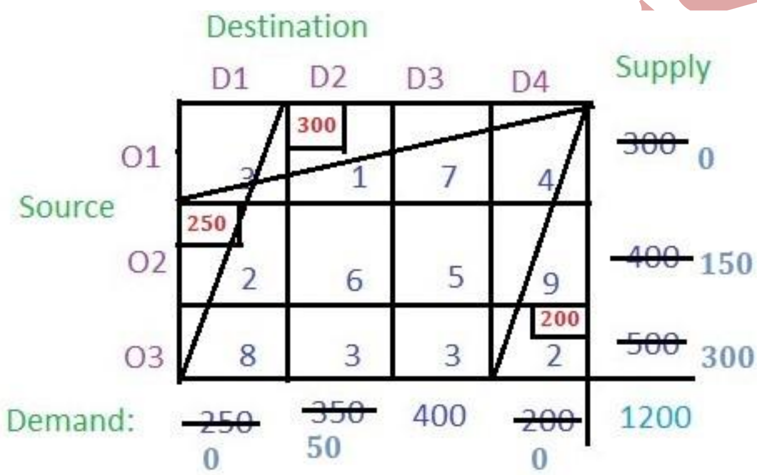
Destination

| Source | D1 | D2 | D3 | D4 | Supply |
|---|---|---|---|---|---|
| O1 | ~~3~~ | ~~300 / 1~~ | ~~7~~ | ~~4~~ | ~~300~~ 0 |
| O2 | 2 | 6 | 5 | 9 | 400 |
| O3 | 8 | 3 | 3 | 2 | 500 |
| Demand: | 250 | ~~350~~ 50 | 400 | 200 | 1200 |

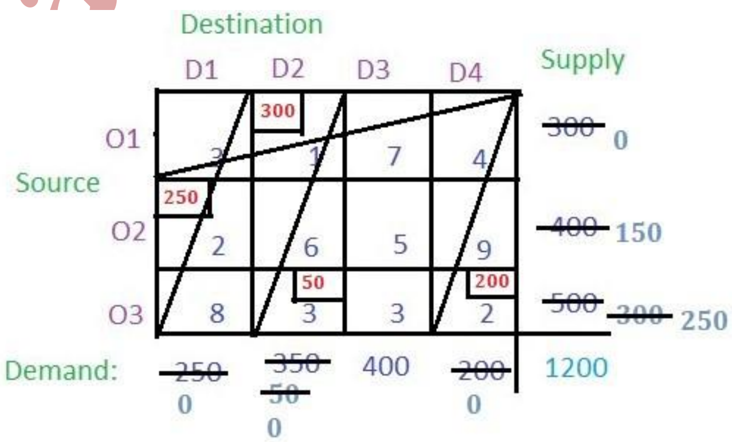
Now find the cell with the least cost among the remaining cells. There are two cells with the least cost i.e. **(O2, D1)** and **(O3, D4)** with cost **2**. Lets select **(O2, D1)**. Now find the demand and supply for the respective cell and allocate the minimum among them to the cell and cancel the row or column whose supply or demand becomes **0** after allocation.



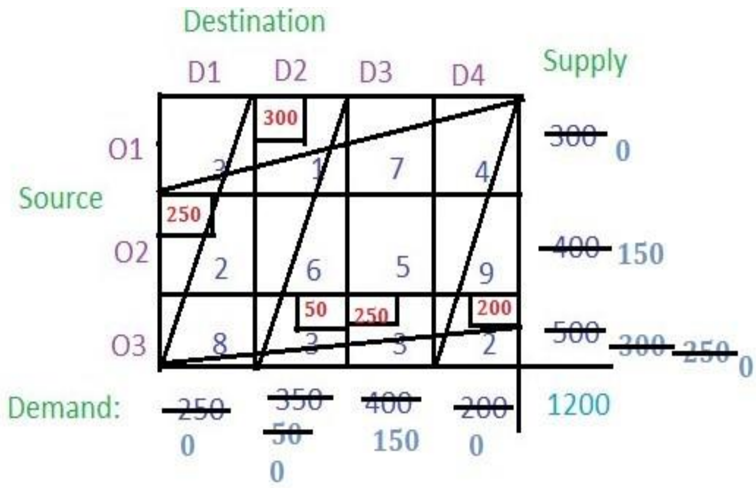

Now the cell with the least cost is **(O3, D4)** with cost **2**. Allocate this cell with **200** as the demand is smaller than the supply. So the column gets canceled.



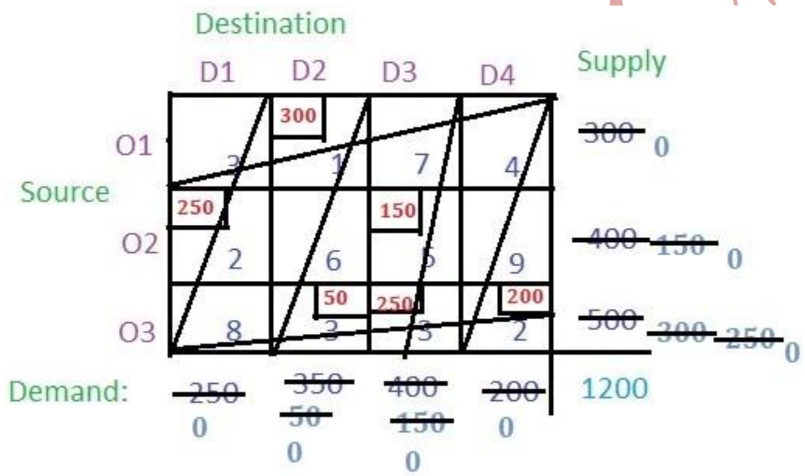

There are two cells among the unallocated cells that have the least cost. Choose any at random say **(O3, D2)**. Allocate this cell with a minimum among the supply from the respective row and the demand of the respective column. Cancel the row or column with zero value.




Now the cell with the least cost is **(O3, D3)**. Allocate the minimum of supply and demand and cancel the row or column with zero value.




The only remaining cell is **(O2, D3)** with cost **5** and its supply is **150** and demand is **150** i.e. demand and supply both are equal. Allocate it to this cell.




Now just multiply the cost of the cell with their respective allocated values and add all of them to get the basic solution i.e. **(300 * 1) + (25 * 2) + (150 * 5) + (50 * 3) + (250 * 3) + (200 * 2) = 2400**

### c) Transportation Problem: (Vogel's Approximation Method)

The **North-West Corner** method and the **Least Cost Cell** method has been discussed in the previous session. In this session, the **Vogel's Approximation** method will be discussed.

Destination

| Source | D1 | D2 | D3 | D4 | Supply |
|---|---|---|---|---|---|
| O1 | 3 | 1 | 7 | 4 | 300 |
| O2 | 2 | 6 | 5 | 9 | 400 |
| O3 | 8 | 3 | 3 | 2 | 500 |
| Demand: | 250 | 350 | 400 | 200 | 1200 |

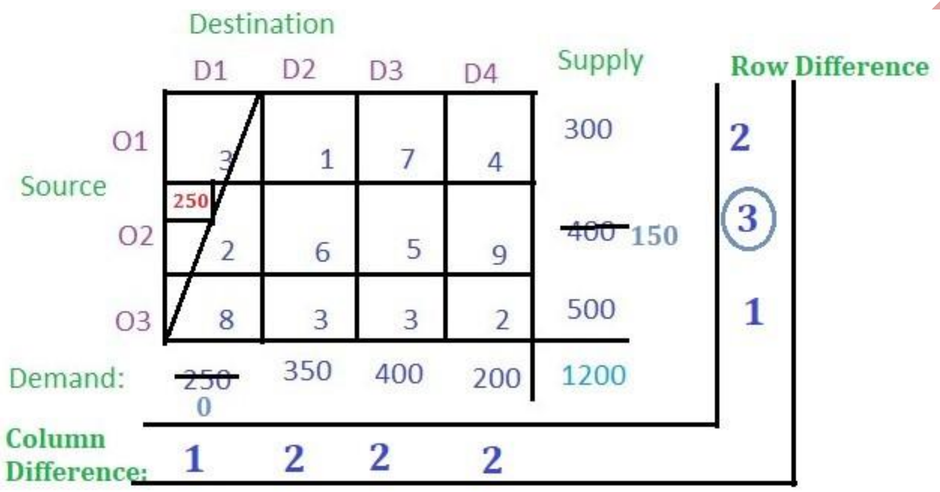
**Solution:**

- For each row find the least value and then the second least value and take the absolute difference of these two least values and write it in the corresponding row difference as shown in the image below. In row **O1**, **1** is the least value and **3** is the second least value and their absolute difference is **2**. Similarly, for row **O2** and **O3**, the absolute differences are **3** and **1** respectively.
- For each column find the least value and then the second least value and take the absolute difference of these two least values then write it in the corresponding column difference as shown in the figure. In column **D1**, **2** is the least value and **3** is the second least value and their absolute difference is **1**. Similarly, for column **D2**, **D3**and **D3**, the absolute differences are **2**, **2** and **2** respectively.

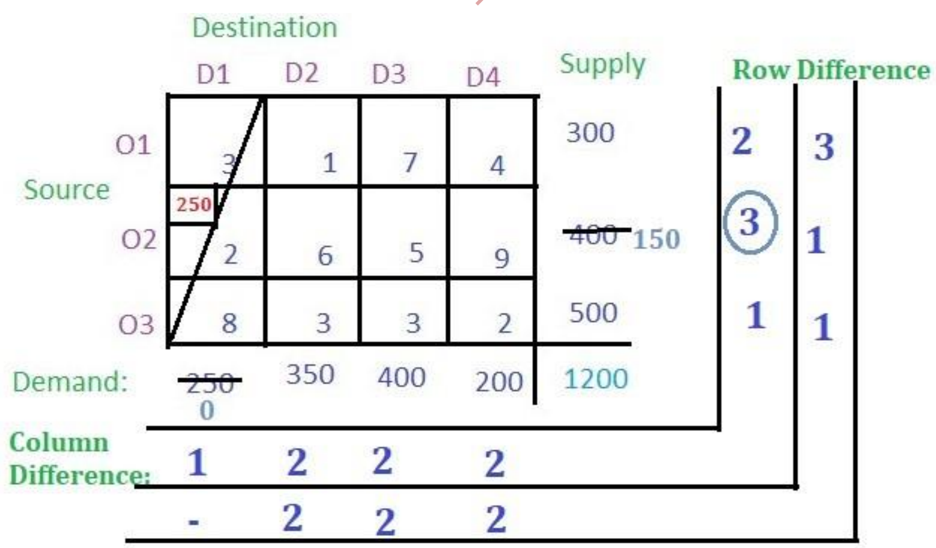
Destination

| Source | D1 | D2 | D3 | D4 | Supply | Row Difference |
|---|---|---|---|---|---|---|
| O1 | 3 | 1 | 7 | 4 | 300 | 2 |
| O2 | 2 | 6 | 5 | 9 | 400 | 3 |
| O3 | 8 | 3 | 3 | 2 | 500 | 1 |
| Demand: | 250 | 350 | 400 | 200 | 1200 | |
| Column Difference: | 1 | 2 | 2 | 2 | | |

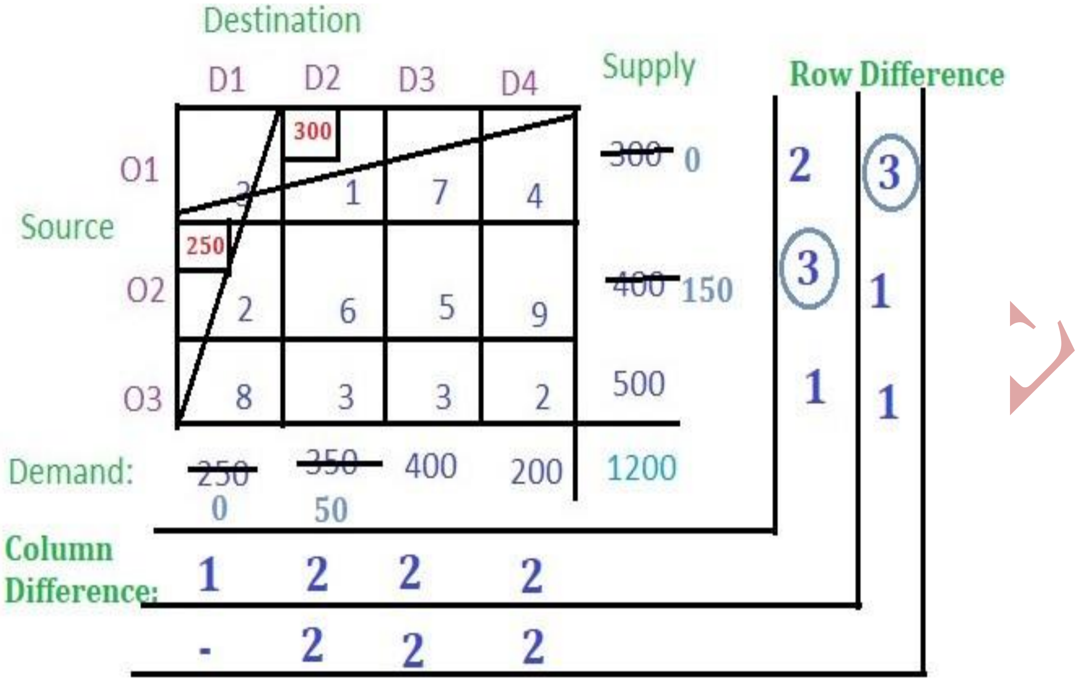
- These value of row difference and column difference are also called as penalty. Now select the maximum penalty. The maximum penalty is **3** i.e. row **O2**. Now find the cell with the least cost in row **O2** and allocate the minimum among the supply of the respective row and the demand of the respective column. Demand is smaller than the supply so allocate the column's demand i.e. **250** to the cell. Then cancel the column **D1**.



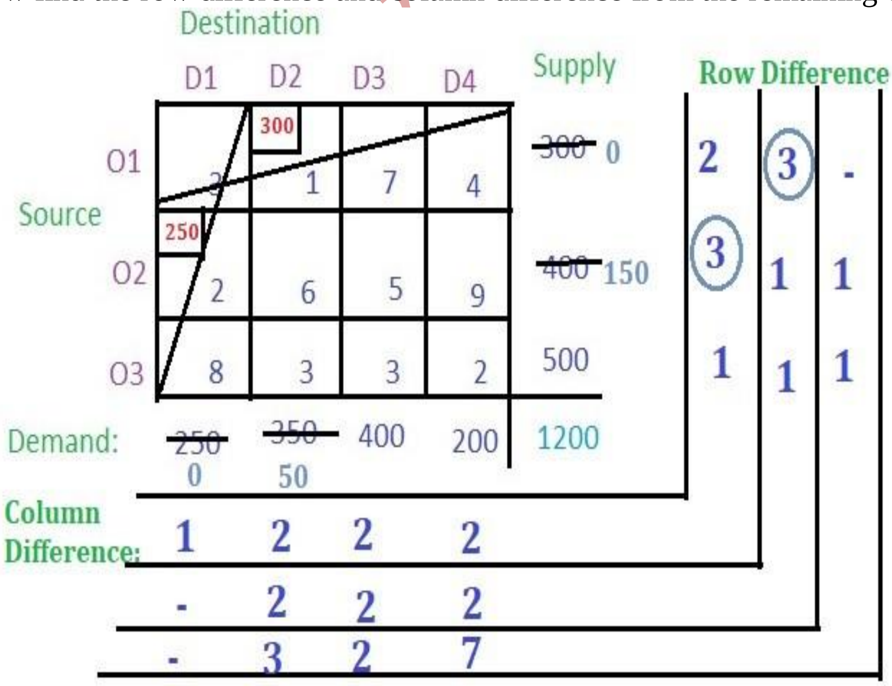

- From the remaining cells, find out the row difference and column difference.



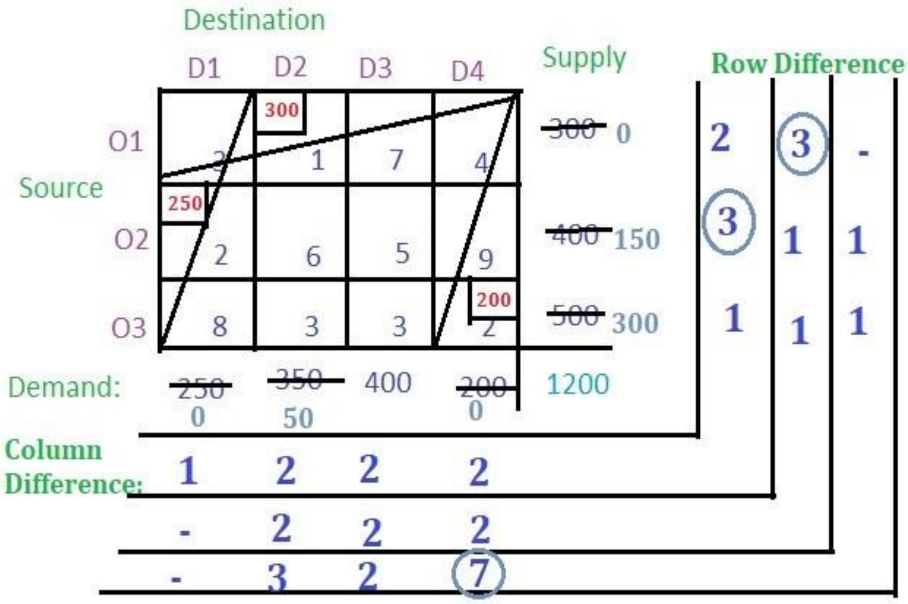

- Again select the maximum penalty which is **3** corresponding to row **O1**. The least-cost cell in row **O1** is **(O1, D2)** with cost **1**. Allocate the minimum among supply and demand from the respective row and column to the cell. Cancel the row or column with zero value.



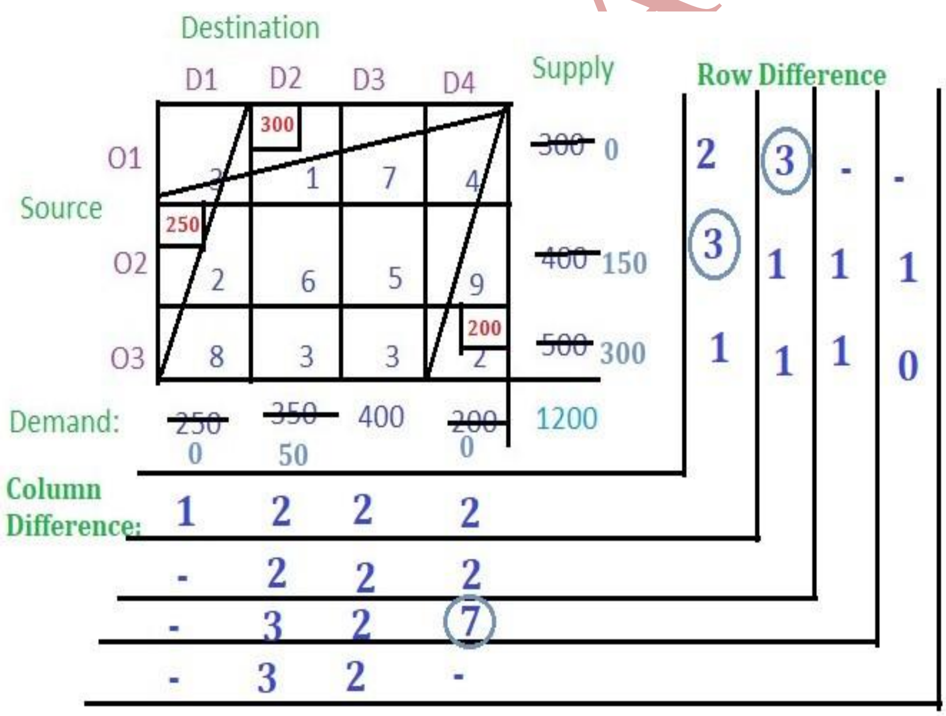

- Now find the row difference and column difference from the remaining cells.



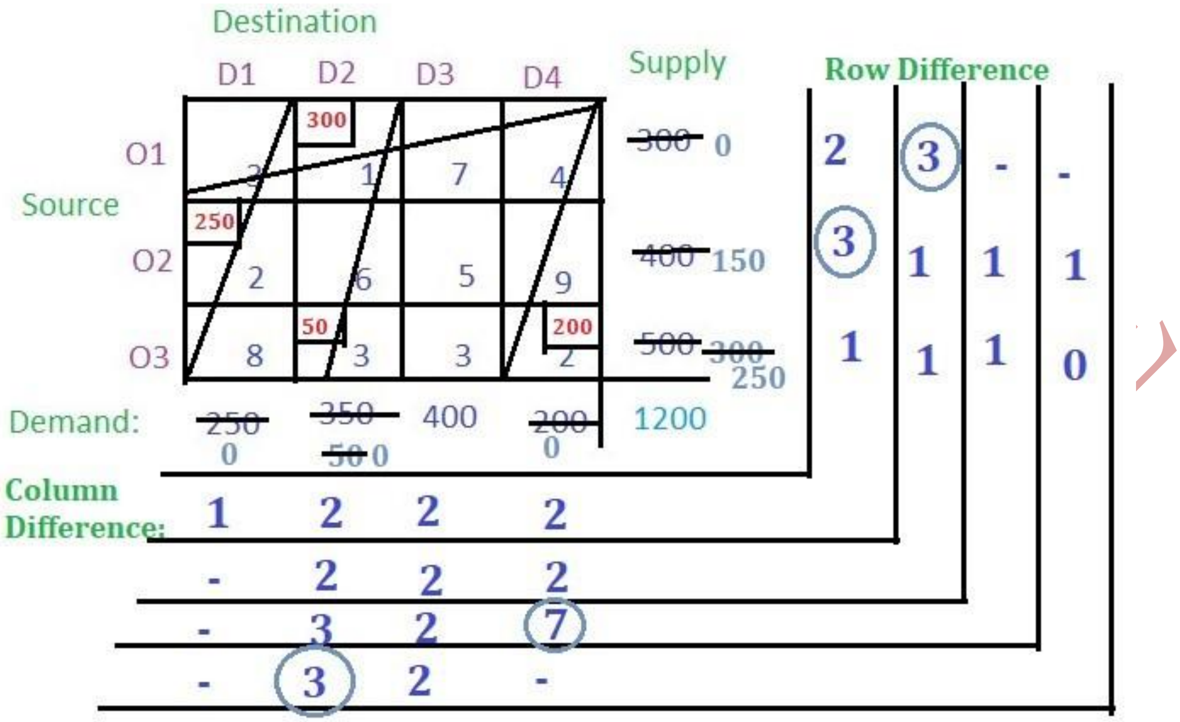

- Now select the maximum penalty which is **7** corresponding to column **D4**. The least cost cell in column **D4** is **(O3, D4)** with cost **2**. The demand is smaller than the supply for cell **(O3, D4)**. Allocate **200** to the cell and cancel the column.



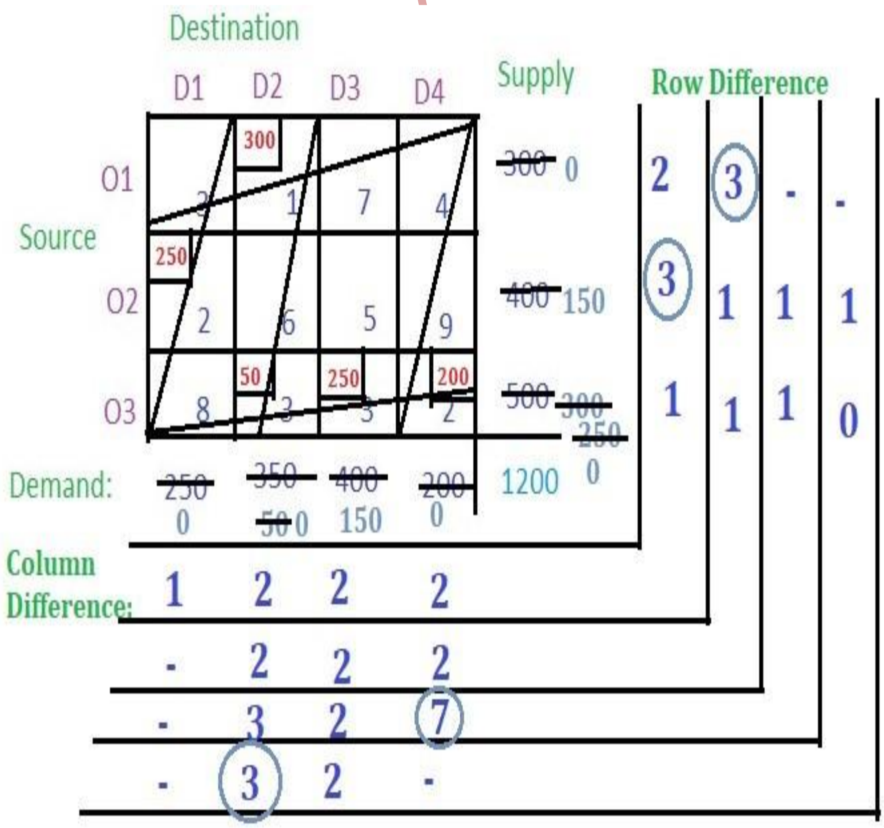

- Find the row difference and the column difference from the remaining cells.



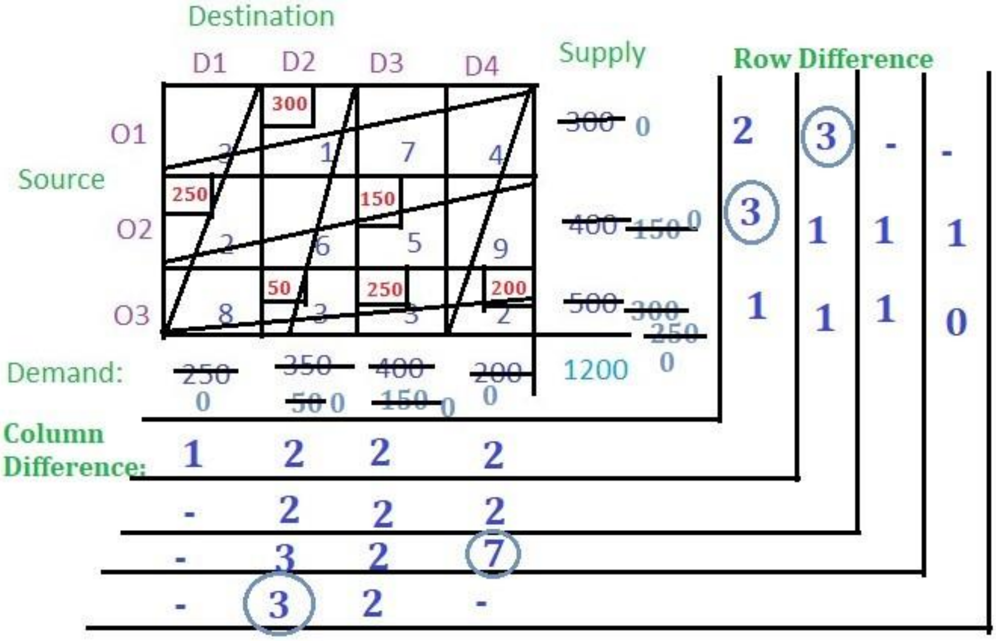

- Now the maximum penalty is **3** corresponding to the column **D2**. The cell with the least value in **D2** is **(O3, D2)**. Allocate the minimum of supply and demand and cancel the column.




- Now there is only one column so select the cell with the least cost and allocate the value.



- Now there is only one cell so allocate the remaining demand or supply to the cell




- No balance remains. So multiply the allocated value of the cells with their corresponding cell cost and add all to get the final cost i.e. **(300 * 1) + (250 * 2) + (50 * 3) + (250 * 3) + (200 * 2) + (150 * 5) = 2850**

### d) Transportation Problem: Unbalanced problem

In this session, the method to solve the unbalanced transportation problem will be discussed.

Below transportation problem is an unbalanced transportation problem.

Destination

| Source | D1 | D2 | D3 | D4 | D5 | Supply($S_i$) |
|---|---|---|---|---|---|---|
| O1 | 5 | 1 | 8 | 7 | 5 | 15 |
| O2 | 3 | 9 | 6 | 7 | 8 | 25 |
| O3 | 4 | 2 | 7 | 6 | 5 | 42 |
| O4 | 7 | 11 | 10 | 4 | 9 | 35 |
| Demand($d_j$): | 30 | 20 | 15 | 10 | 20 | 95 \| 117 |

The problem is unbalanced because the sum of all the supplies i.e. **O1**, **O2**, **O3** and **O4** is not equal to the sum of all the demands i.e. **D1**, **D2**, **D3**, **D4** and **D5**.

**Solution:**

In this type of problem, the concept of a dummy row or a dummy column will be used. As in this case, since the supply is more than the demand so a dummy demand column will be added and a demand of (total supply – total demand) will be given to that column i.e. **117 – 95 = 22** as shown in the image below. If demand were more than the supply then a dummy supply row would have been added.

Destination

| Source | D1 | D2 | D3 | D4 | D5 | | Supply($S_i$) |
|---|---|---|---|---|---|---|---|
| O1 | 5 | 1 | 8 | 7 | 5 | 0 | 15 |
| O2 | 3 | 9 | 6 | 7 | 8 | 0 | 25 |
| O3 | 4 | 2 | 7 | 6 | 5 | 0 | 42 |
| O4 | 7 | 11 | 10 | 4 | 9 | 0 | 35 |
| Demand($d_j$): | 30 | 20 | 15 | 10 | 20 | 22 | 117 \| 117 |

Now that the problem has been updated to a balanced transportation problem, it can be solved using any one of the following methods to solve a balanced transportation problem as discussed in the earlier posts:

1. NorthWest Corner Method
2. Least Cost Cell Method
3. Vogel's Approximation Method

# Optimal solution: MODI Method – UV Method

There are two phases to solve the transportation problem. In the first phase, the initial basic feasible solution has to be found and the second phase involves optimization of the initial basic feasible solution that was obtained in the first phase. There are three methods for finding an initial basic feasible solution,

1. NorthWest Corner Method
2. Least Cost Cell Method
3. Vogel's Approximation Method

Will discuss how to optimize the initial basic feasible solution through an explained example. Consider the below transportation problem.

Destination

| Source | D1 | D2 | D3 | D4 | Supply($S_i$) |
|---|---|---|---|---|---|
| O1 | 3 | 1 | 7 | 4 | 250 |
| O2 | 2 | 6 | 5 | 9 | 350 |
| O3 | 8 | 3 | 3 | 2 | 400 |
| Demand($D_j$): | 200 | 300 | 350 | 150 | |

**Solution:**

**Step 1:** Check whether the problem is balanced or not.

If the total sum of all the supply from sources **O1**, **O2**, and **O3** is equal to the total sum of all the demands for destinations **D1**, **D2**, **D3** and **D4** then the transportation problem is a balanced transportation problem.

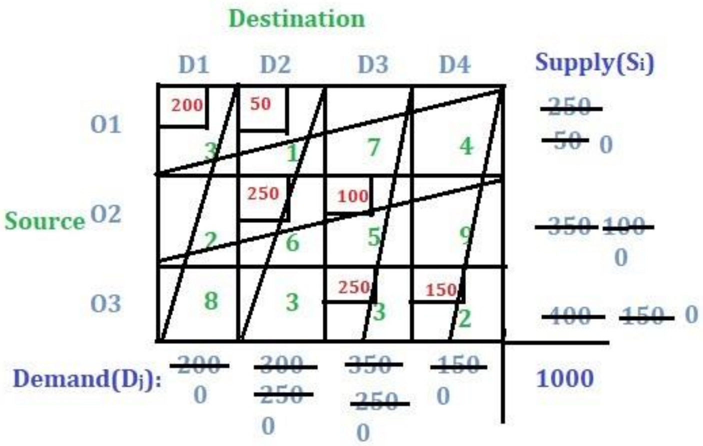
Destination

| Source | D1 | D2 | D3 | D4 | Supply($S_i$) |
|---|---|---|---|---|---|
| O1 | 3 | 1 | 7 | 4 | 250 |
| O2 | 2 | 6 | 5 | 9 | 350 |
| O3 | 8 | 3 | 3 | 2 | 400 |
| Demand($D_j$): | 200 | 300 | 350 | 150 | 1000 |

**Note:** If the problem is not unbalanced then the concept of a dummy row or a dummy column to transform the unbalanced problem to balanced can be followed as discussed.

**Step 2:** Finding the initial basic feasible solution.

Any of the three aforementioned methods can be used to find the initial basic feasible solution. Here, NorthWest Corner Method will be used. And according to the NorthWest Corner Method this is the final initial basic feasible solution:




Now, the total cost of transportation will be **(200 * 3) + (50 * 1) + (250 * 6) + (100 * 5) + (250 * 3) + (150 * 2) = 3700**.

**Step 3:** U-V method to optimize the initial basic feasible solution.

The following is the initial basic feasible solution:

| | | | |
|---|---|---|---|
| 200 3 | 50 1 | 7 | 4 |
| 2 | 250 6 | 100 5 | 9 |
| 8 | 3 | 250 3 | 150 2 |

– For U-V method the values $u_i$ and $v_j$ have to be found for the rows and the columns respectively. As there are three rows so three $u_i$ values have to be found i.e. $u_1$ for the first row, $u_2$ for the second row and $u_3$ for the third row.

Similarly, for four columns four $\mathbf{v}_j$ values have to be found i.e. $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$. Check the image below:

| | V1= | V2= | V3= | V4= |
|---|---|---|---|---|
| u1 = | 200 / 3 | 50 / 1 | 7 | 4 |
| u2 = | 2 | 250 / 6 | 100 / 5 | 9 |
| u3 = | 8 | 3 | 250 / 3 | 150 / 2 |

There is a separate formula to find $\mathbf{u}_i$ and $\mathbf{v}_j$,

$\mathbf{u}_i + \mathbf{v}_j = \mathbf{C}_{ij}$ where $\mathbf{C}_{ij}$ is the cost value only for the allocated cell.

Before applying the above formula we need to check whether **m + n – 1 is equal to the total number of allocated cells** or not where **m** is the total number of rows and **n** is the total number of columns.

In this case m = 3, n = 4 and total number of allocated cells is 6 so m + n – 1 = 6. The case when m + n – 1 is not equal to the total number of allocated cells will be discussed in the later posts.

Now to find the value for u and v we assign any of the three u or any of the four v as 0. Let we assign $\mathbf{u}_1 = \mathbf{0}$ in this case. Then using the above formula we will get $\mathbf{v}_1 = \mathbf{3}$ as $\mathbf{u}_1 + \mathbf{v}_1 = \mathbf{3}$ (i.e. $\mathbf{C}_{11}$) and $\mathbf{v}_2 = \mathbf{1}$ as $\mathbf{u}_1 + \mathbf{v}_2 = \mathbf{1}$ (i.e. $\mathbf{C}_{12}$). Similarly, we have got the value for $\mathbf{v}_2 = \mathbf{3}$ so we get the value for $\mathbf{u}_2 = \mathbf{5}$ which implies $\mathbf{v}_3 = \mathbf{0}$. From the value of $\mathbf{v}_3 = \mathbf{0}$ we get $\mathbf{u}_3 = \mathbf{3}$which implies $\mathbf{v}_4 = \mathbf{-1}$.

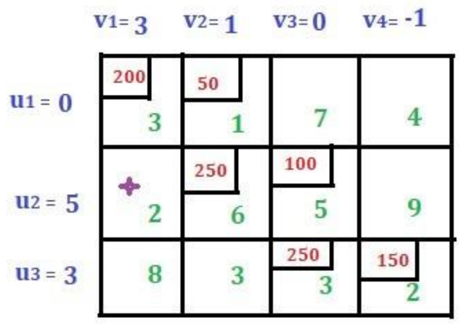
| | v1=-3 | v2=1 | v3=0 | v4=-1 |
|---|---|---|---|---|
| u1=0 | 3 | 250 / 1 | 7 | 4 |
| u2=5 | 200 / 2 | 50 / 6 | 100 / 5 | 9 |
| u3=3 | 8 | 3 | 250 / 3 | 150 / 2 |

Now, compute penalties using the formula $\mathbf{P}_{ij} = \mathbf{u}_i + \mathbf{v}_j - \mathbf{C}_{ij}$ only for unallocated cells. We have two unallocated cells in the first row, two in the second row and two in the third row. Let's compute this one by one.

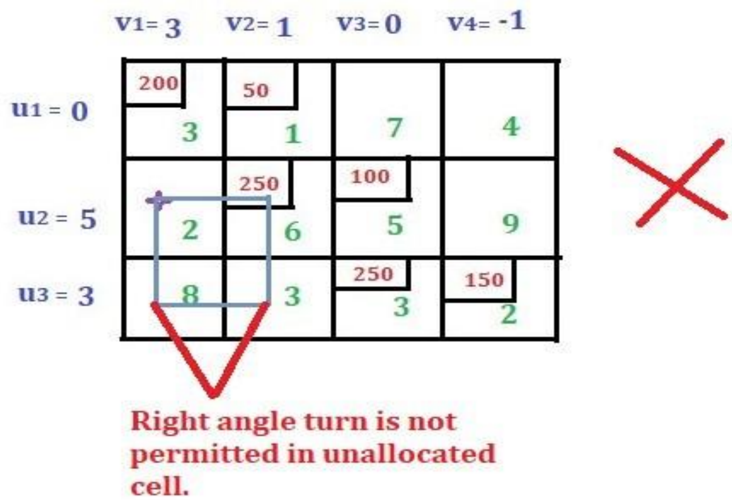
1. For $\mathbf{C_{13}}$, $\mathbf{P_{13} = 0 + 0 - 7 = -7}$ (here $\mathbf{C_{13} = 7}$, $\mathbf{u_1 = 0}$ and $\mathbf{v_3 = 0}$)
2. For $\mathbf{C_{14}}$, $\mathbf{P_{14} = 0 + (-1) -4 = -5}$
3. For $\mathbf{C_{21}}$, $\mathbf{P_{21} = 5 + 3 - 2 = 6}$
4. For $\mathbf{C_{24}}$, $\mathbf{P_{24} = 5 + (-1) - 9 = -5}$
5. For $\mathbf{C_{31}}$, $\mathbf{P_{31} = 3 + 3 - 8 = -2}$
6. For $\mathbf{C_{32}}$, $\mathbf{P_{32} = 3 + 1 - 3 = 1}$

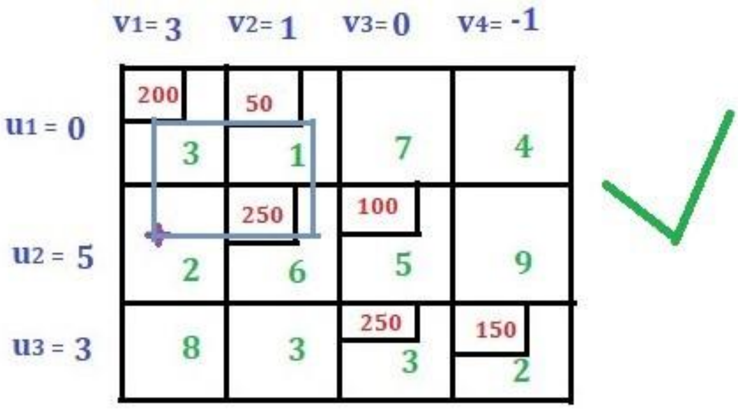
**The Rule:** If we get all the penalties value as zero or negative values that mean the optimality is reached and this answer is the final answer. But if we get any positive value means we need to proceed with the sum in the next step.

Now find the maximum positive penalty. Here the maximum value is 6 which corresponds to $\mathbf{C_{21}}$ cell. Now this cell is new basic cell. This cell will also be included in the solution.

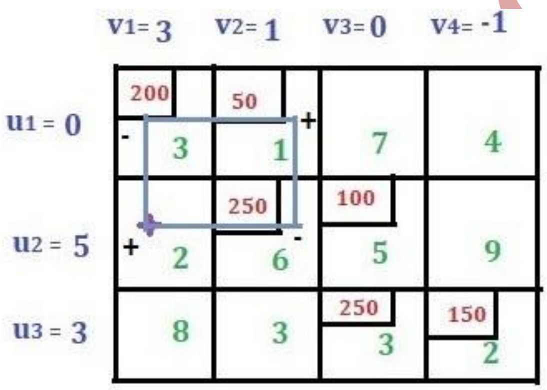



**The rule for drawing closed-path or loop.** Starting from the new basic cell draw a closed-path in such a way that the right angle turn is done only at the allocated cell or at the new basic cell.

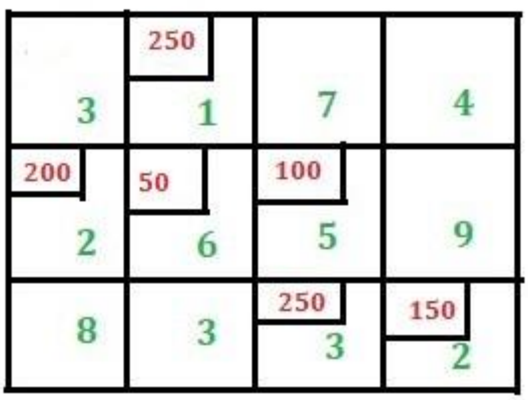





Assign alternate plus-minus sign to all the cells with right angle turn (or the corner) in the loop with plus sign assigned at the new basic cell.




Consider the cells with a negative sign. Compare the allocated value (i.e. 200 and 250 in this case) and select the minimum (i.e. select 200 in this case). Now subtract 200 from the cells with a minus sign and add 200 to the cells with a plus sign. And draw a new iteration. The work of the loop is over and the new solution looks as shown below.



Check the total number of allocated cells is equal to (m + n – 1). Again find u values and v values using the formula $\mathbf{u_i + v_j = C_{ij}}$ where $\mathbf{C_{ij}}$ is the cost value only for allocated cell. Assign $\mathbf{u_1 = 0}$ then we get $\mathbf{v_2 = 1}$. Similarly, we will get following values for $\mathbf{u_i}$ and $\mathbf{v_j}$.

| | $v_1=-3$ | $v_2=1$ | $v_3=0$ | $v_4=-1$ |
|---|---|---|---|---|
| $u_1=0$ | 3 | [250] 1 | 7 | 4 |
| $u_2=5$ | [200] 2 | [50] 6 | [100] 5 | 9 |
| $u_3=0$ | 8 | 3 | [250] 3 | [150] 2 |

Find the penalties for all the unallocated cells using the formula $\mathbf{P_{ij} = u_i + v_j - C_{ij}}$.

1. For $\mathbf{C_{11}}$, $\mathbf{P_{11} = 0 + (-3) - 3 = -6}$
2. For $\mathbf{C_{13}}$, $\mathbf{P_{13} = 0 + 0 - 7 = -7}$
3. For $\mathbf{C_{14}}$, $\mathbf{P_{14} = 0 + (-1) - 4 = -5}$
4. For $\mathbf{C_{24}}$, $\mathbf{P_{24} = 5 + (-1) - 9 = -5}$
5. For $\mathbf{C_{31}}$, $\mathbf{P_{31} = 0 + (-3) - 8 = -11}$
6. For $\mathbf{C_{32}}$, $\mathbf{P_{32} = 3 + 1 - 3 = 1}$

There is one positive value i.e. 1 for $\mathbf{C_{32}}$. Now this cell becomes new basic cell.

| | $v_1=-3$ | $v_2=1$ | $v_3=0$ | $v_4=-1$ |
|---|---|---|---|---|
| $u_1=0$ | 3 | [250] 1 | 7 | 4 |
| $u_2=5$ | [200] 2 | [50] 6 | [100] 5 | 9 |
| $u_3=3$ | 8 | + 3 | [250] 3 | [150] 2 |

Now draw a loop starting from the new basic cell. Assign alternate plus and minus sign with new basic cell assigned as a plus sign.

| | $v_1=-3$ | $v_2=1$ | $v_3=0$ | $v_4=-1$ |
|---|---|---|---|---|
| $u_1=0$ | 3 | 250<br>1 | 7 | 4 |
| $u_2=5$ | 200<br>2 | 50<br>- 6 | 100<br>+5 | 9 |
| $u_3=3$ | 8 | + 3 | 250<br>- 3 | 150<br>2 |

Select the minimum value from allocated values to the cell with a minus sign. Subtract this value from the cell with a minus sign and add to the cell with a plus sign. Now the solution looks as shown in the image below:

| | | | |
|---|---|---|---|
| 3 | 250<br>1 | 7 | 4 |
| 200<br>2 | 6 | 150<br>5 | 9 |
| 8 | 50<br>3 | 200<br>3 | 150<br>2 |

Check if the total number of allocated cells is equal to (m + n – 1). Find u and v values as above.

| | $v_1=-2$ | $v_2=1$ | $v_3=1$ | $v_4=0$ |
|---|---|---|---|---|
| $u_1=0$ | 3 | 250<br>1 | 7 | 4 |
| $u_2=4$ | 200<br>2 | 6 | 150<br>5 | 9 |
| $u_3=2$ | 8 | 50<br>3 | 200<br>3 | 150<br>2 |

Now again find the penalties for the unallocated cells as above.

1. For $\mathbf{P_{11} = 0 + (-2) - 3 = -5}$
2. For $\mathbf{P_{13} = 0 + 1 - 7 = -6}$
3. For $\mathbf{P_{14} = 0 + 0 - 4 = -4}$
4. For $\mathbf{P_{22} = 4 + 1 - 6 = -1}$
5. For $\mathbf{P_{24} = 4 + 0 - 9 = -5}$
6. For $\mathbf{P_{31} = 2 + (-2) - 8 = -8}$

All the penalty values are negative values. So the optimality is reached.

Now, find the total cost i.e. **(250 * 1) + (200 * 2) + (150 * 5) + (50 * 3) + (200 * 3) + (150 * 2) = 2450**

**Transportation Problem: Degeneracy in Transportation Problem**

This session will discuss degeneracy in transportation problem through an explained example.

Destination

| Source | D1 | D2 | D3 | D4 | D5 | Supply($S_i$) |
|---|---|---|---|---|---|---|
| S1 | 10 | 2 | 3 | 15 | 9 | 35 |
| S2 | 5 | 10 | 15 | 2 | 4 | 40 |
| S3 | 15 | 5 | 14 | 7 | 15 | 20 |
| S4 | 20 | 15 | 13 | 25 | 8 | 30 |
| Demand($D_j$): | 20 | 20 | 40 | 10 | 35 | |

**Solution:**

This problem is balanced transportation problem as total supply is equal to total demand.

Destination

| Source | D1 | D2 | D3 | D4 | D5 | Supply($S_i$) |
|---|---|---|---|---|---|---|
| S1 | 10 | 2 | 3 | 15 | 9 | 35 |
| S2 | 5 | 10 | 15 | 2 | 4 | 40 |
| S3 | 15 | 5 | 14 | 7 | 15 | 20 |
| S4 | 20 | 15 | 13 | 25 | 8 | 30 |
| Demand($D_j$): | 20 | 20 | 40 | 10 | 35 | 125 |

**Initial basic feasible solution:**

Least Cost Cell Method will be used here to find the initial basic feasible solution. One can also use NorthWest Corner Method or Vogel's Approximation Method to find the initial basic feasible solution.

Using Least Cost Cell Method we get the following solution.

Destination

| Source | D1 | D2 | D3 | D4 | D5 | Supply($S_i$) |
|---|---|---|---|---|---|---|
| S1 | 10 | (20) 2 | (15) 3 | 15 | 9 | 35 |
| S2 | 5 | 10 | 15 | (10) 2 | (30) 4 | 40 |
| S3 | (20) 15 | 5 | 14 | 7 | 15 | 20 |
| S4 | 20 | 15 | (25) 13 | 25 | (5) 8 | 30 |
| Demand($D_j$): | 20 | 20 | 40 | 10 | 35 | 125 |

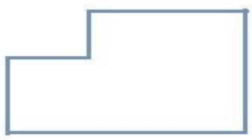
**Optimization of the solution using U-V Method:**

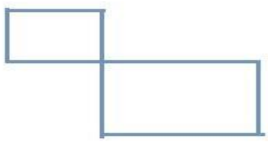
Check whether m + n – 1 = total number of allocated cells. In this case m + n – 1 = 4 + 5 – 1 = 8 where as total number of allocated cells are 7, hence this is the case of degeneracy in transportation problem. So in this case we convert the necessary number (in this case it is m + n – 1 – total number of allocated cells i.e. 8 – 7 = 1) of unallocated cells into allocated cells to satisfy the above condition.

**Steps to convert unallocated cells into allocated cells:**

- Start from the least value of the unallocated cell.
- Check the loop formation one by one.
- There should be no closed-loop formation.
- Select that loop as a new allocated cell and assign a value 'e'.

The closed loop can be in any form but all the turning point should be only at allocated cell or at the cell from the loop is started.





There are 13 unallocated cells. Select the least value (i.e. 5 in this case) from unallocated cells. There are two 5s here so you can select randomly any one. Lets select the cell with star marked.

Destination

| Source | D1 | D2 | D3 | D4 | D5 | Supply($S_i$) |
|---|---|---|---|---|---|---|
| S1 | 10 | (20) 2 | (15) 3 | 15 | 9 | 35 |
| S2 | 5 | 10 | 15 | (10) 2 | (30) 4 | 40 |
| S3 | (20) 15 | + 5 | 14 | 7 | 15 | 20 |
| S4 | 20 | 15 | (25) 13 | 25 | (5) 8 | 30 |
| Demand($D_j$): | 20 | 20 | 40 | 10 | 35 | 125 |

Check if there is any closed-loop formation starting from this cell. If a closed-loop is drawn from this cell following the condition for closed-loop then it can be observed that this cell cannot be reached to complete the closed-loop. So this cell will be selected and assigned a random value 'e'.

Destination

| Source | D1 | D2 | D3 | D4 | D5 | Supply($S_i$) |
|---|---|---|---|---|---|---|
| S1 | 10 | (20) 2 | (15) 3 | 15 | 9 | 35 |
| S2 | 5 | 10 | 15 | (10) 2 | (30) 4 | 40 |
| S3 | (20) 15 | + (e) 5 | 14 | 7 | 15 | 20 |
| S4 | 20 | 15 | (25) 13 | 25 | (5) 8 | 30 |
| Demand($D_j$): | 20 | 20 | 40 | 10 | 35 | 125 |

**Note:** If the closed loop would have been formed from that cell then we would try another cell with least value and do the same procedure and check whether closed loop is possible or not.

Now total number of allocated cells becomes 8 and m + n – 1 = 4 + 5 – 1 = 8. Now this solution can be optimized using U-V method. We get the below solution after performing optimization using U-V method.

| | | | | |
|---|---|---|---|---|
| 10 | e 2 | 35 3 | 15 | 9 |
| 20 5 | 10 | 15 | 10 2 | 10 4 |
| 15 | 20+e 5 | 14 | 7 | 15 |
| 20 | 15 | 5 13 | 25 | 25 8 |

The presence of two ‘e’ in the final solution means after doing some iterations during optimization, the condition for degeneracy will be met once again.

While finding the total cost, just leave the ‘e’ and multiply the allocated value with its cell’s cost value and add all of them. So, the transportation cost is **(35 * 3) + (20 * 5) + (10 * 2) + (10 * 4) + (20 * 5) + (5 * 13) + (25 * 8) = 630**.

# ASSIGNMENT PROBLEMS

## INTRODUCTION

In Block 1 of this course, we have discussed the basic concepts elated to Linear Programming Problems and the Simplex method for solving them. The Transportation Problem was also discussed in Block 1. In this unit, we explain the Assignment problem and discuss various methods for solving it.

The assignment problem deals with allocating various resources (items) to various activities (receivers) on a one to one basis, i.e., the number of operations are to be assigned to an equal number of operators where each operator performs only one operation. For example, suppose an accounts officer has 4 subordinates and 4 tasks. The subordinates differ in efficiency and take different time to perform each task. If one task is to be assigned to one person in such a way that the total person hours are minimised, the problem is called an **assignment problem**. Though the assignment problem is a special case of transportation problem, it is not solved using the methods described in Unit 4. We use another method called the Hungarian method for solving an assignment problem. It is shorter and easier compared to any method of finding the optimal solution of a transportation problem. In this unit, we discuss various types of assignment problems, including travelling salesman problem and apply the Hungarian method for solving these problems.

In the next unit, we shall discuss the fundamental structure and operating characteristics of a queueing system and explain a single server M/M/1 queueing model with Poisson input and exponential service time.

### Objectives

After studying this unit, you should be able to:

- formulate an assignment problem;
- determine the optimal solutions of assignment problems using the Hungarian method;
- obtain the solutions for special cases of assignment problems, i.e, maximisation problem, unbalanced assignment problem, alternative optimal solutions and restriction on assignments; and
- Solve the travelling salesman problem as an assignment problem.

An assignment problem may be considered as a special type of transportation problem in which the number of sources and destinations are equal. The capacity of each source as well as the requirement of each destination is taken as 1. In the case of an assignment problem, the given matrix must necessarily be a square matrix which is not the condition for a transportation problem.

Suppose there are n persons and n jobs and the assignment of jobs has to be done on a one-to-one basis. This assignment problem can be stated in the form of an n×n matrix of real numbers (known as the **cost matrix**) as given in the following table:

| | Job | | | | | |
|---|---|---|---|---|---|---|
| Person | 1 | 2 | ... | j | ... | n |
| 1 | $C_{11}$ | $C_{12}$ | ... | $C_{1j}$ | ... | $C_{1n}$ |
| 2 | $C_{21}$ | $C_{22}$ | ... | $C_{2j}$ | ... | $C_{2n}$ |
| . | . | | | | | |
| . | . | | | | | |
| i | $C_{i1}$ | $C_{i2}$ | ... | $C_{ij}$ | ... | $C_{in}$ |
| . | . | | | | | |
| . | . | | | | | |
| n | $C_{n1}$ | $C_{n2}$ | ... | $C_{nj}$ | ... | $C_{nn}$ |

where $c_{ij}$ represents the amount of time taken by $i^{th}$ person to complete the $j^{th}$ job. Let $x_{ij}$ denote the $j^{th}$ job assigned to the $i^{th}$ person. Then, mathematically, the assignment problem can be stated as follows:

$$\text{Minimise } Z = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij}, \quad \text{where } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, n.$$

subject to

1 if the $i^{th}$ person isassigned the $j^{th}$ job

0 if the $i^{th}$ person is not assigned the j job

$x_{i1} + x_{i2} + \ldots + x_{in} = 1, \quad i = 1, 2, \ldots, n$ (one job is done by the $i^{th}$ person)

$x_{1j} + x_{2j} + \ldots + x_{nj} = 1, \; j = 1, 2, \ldots n$ (only one person is assigned the $j^{th}$ job)

The constant $c_{ij}$ in the above problem represents time. It may be cost or some other parameter which is to be minimised in the assignment problem under consideration.

Note that an assignment problem is a special type of transportation problem and may be solved as one. However, we use another method known as the Hungarian method for solving it. This method is shorter and easier compared to any other method of finding the optimal solution of a transportation problem. Let us explain the Hungarian method of finding the optimal solution of an assignment problem.

### Hungarian Method of Solving an Assignment Problem

The steps for obtaining an optimal solution of an assignment problem are as follows:

1. Check whether the given matrix is square. If not, make it square by adding a suitable number of dummy rows (or columns) with 0 cost/time elements.
2. Locate the smallest cost element in each row of the cost matrix. Subtract the smallest element of each row from every element of that row.
3. In the resulting cost matrix, locate the smallest element in each column and subtract the smallest element of each column from every element of that column.
4. In the resulting matrix, search for an optimum assignment as follows:

   i) Examine the rows successively until a row with exactly one zero is found. Draw a rectangle around this zero (as 0) and cross out all other zeroes in the corresponding column. Proceed in this manner until all the rows have been examined. If there is more than one zero in any row, do not touch that row; pass on to the next row.

   ii) Repeat step (i) above for the columns of the resulting cost matrix.

   iii) If a row or column of the reduced matrix contains more than one zeroes, arbitrarily choose a row or column having the minimum number of zeroes. Arbitrarily select any zero in the row or column so chosen. Draw a rectangle around it and cross out all the zeroes in the corresponding row and column. Repeat steps (i), (ii) and (iii) until all the zeroes have either been assigned (by drawing a rectangle around them) or crossed.

   iv) If each row and each column of the resulting matrix has one and only one assigned 0, the optimum assignment is made in the cells corresponding to $\boxed{0}$ . The optimum solution of the problem is attained and you can stop here.

Otherwise, go to the next step.

5. Draw the minimum number of horizontal and/or vertical lines through all the zeroes as follows:

   i) Tick mark ( √ ) the rows in which assignment has not been made.

   ii) Tick mark ( √ ) columns, which have zeroes in the marked rows.

   iii) Tick mark ( √ ) rows (not already marked) which have assignments in marked columns. Then tick mark ( √ ) columns, which have zeroes in newly marked rows, if any. Tick mark ( √ ) rows (not already marked), which have assignments in these newly marked columns.

   iv) Draw straight lines through all unmarked rows and marked columns.

6. Revise the cost matrix as follows:

   i) Find the smallest element not covered by any of the lines.

   ii) Subtract this from all the uncovered elements and add it to the elements at the intersection of the two lines.

   iii) Other elements covered by the lines remain unchanged.

7. Repeat the procedure until an optimum solution is attained.

**We now illustrate the procedure with the help of an example.**

**Example 1:** A computer centre has four expert programmers and needs to develop four application programmes. The head of the computer centre, estimates the computer time (in minutes) required by the respective experts to develop the application programmes as follows:

| | | Programmes | | | |
|---|---|---|---|---|---|
| | | **A** | **B** | **C** | **D** |
| **Programmers** | **1** | 120 | 100 | 80 | 90 |
| | **2** | 80 | 90 | 110 | 70 |
| | **3** | 110 | 140 | 120 | 100 |
| | **4** | 90 | 90 | 80 | 90 |

Find the assignment pattern that minimises the time required to develop the application programmes.

**Solution:** Let us subtract the minimum element of each row from every element of that row. Note that the minimum element in the first row is 80. So 80 is to be subtracted from every element of the first row, i.e., from 120, 100, 80 and 90, respectively. As a result, the elements of the first row of the resulting matrix would be 40, 20, 0, 10, respectively. Similarly, we obtain the elements of the other rows of the resulting matrix. Thus, the resulting matrix is:

| | A | B | C | D |
|---|---|---|---|---|
| 1 | 40 | 20 | 0 | 10 |
| 2 | 10 | 20 | 40 | 0 |
| 3 | 10 | 40 | 20 | 0 |
| 4 | 10 | 10 | 0 | 10 |

Let us now subtract the minimum element of each column from every element of that column in the resulting matrix. The minimum element in the first column is 10. So 10 is to be subtracted from every element of the first column, i.e., from 40, 10, 10, and 10, respectively. As a result, the elements of the first column of the resulting matrix are 30, 0, 0, 0, respectively. Similarly, we obtain the elements of the other columns of the resulting matrix. Thus, the resulting matrix is:

| | A | B | C | D |
|---|---|---|---|---|
| 1 | 30 | 10 | 0 | 10 |
| 2 | 0 | 10 | 40 | 0 |
| 3 | 0 | 30 | 20 | 0 |
| 4 | 0 | 0 | 0 | 10 |

Now, starting from first row onward, we draw a rectangle around the 0 in each row having a single zero and cross all other zeroes in the corresponding column. Here, in the very first row we find a single zero. So, we draw a rectangle around it and cross all other zeroes in the corresponding column.

We get

| | A | B | C | D |
|---|---|---|---|---|
| 1 | 30 | 10 | $\boxed{0}$ | 10 |
| 2 | 0 | 10 | 40 | 0 |
| 3 | 0 | 30 | 20 | 0 |
| 4 | 0 | 0 | $\cancel{0}$ | 10 |

In the second, third and fourth row, there is no single zero. Hence, we move column-wise. In the second column, we have a single zero. Hence, we draw a rectangle around it and cross all other zeroes in the corresponding row. We get

| | A | B | C | D |
|---|---|---|---|---|
| 1 | 30 | 10 | $\boxed{0}$ | 10 |
| 2 | 0 | 10 | 40 | 0 |
| 3 | 0 | 30 | 20 | 0 |
| 4 | $\cancel{0}$ | $\boxed{0}$ | $\cancel{0}$ | 10 |

In the matrix above, there is no row or column, which has a single zero. Therefore, we first move row-wise to locate the row having more than one zero. The second row has two zeroes. So, we draw a rectangle arbitrarily around one of these zeroes and cross the other one. Let us draw a rectangle around the zero in the cell (2, A) and cross the zero in the cell (2, D). We cross out the other zeroes in the first column. Note that we could just as well have selected the zero in the cell (2, D), drawn a rectangle around it and crossed all other zeroes. This would have led to an alternative solution.

In this way, we are left with only one zero in every row and column around which a rectangle has been drawn. This means that we have assigned only one operation to one operator. Thus, we get the optimum solution as follows:

| | A | B | C | D |
|---|---|---|---|---|
| 1 | 30 | 10 | $\boxed{0}$ | 10 |
| 2 | $\boxed{0}$ | 10 | 40 | $\cancel{0}$ |
| 3 | $\cancel{0}$ | 30 | 20 | $\boxed{0}$ |
| 4 | $\cancel{0}$ | $\boxed{0}$ | $\cancel{0}$ | 10 |

Note that the assignment of jobs should be made on the basis of the cells corresponding to the zeroes around which rectangles have been drawn. Therefore, the optimum solution for this problem is:

$1 \rightarrow C,\ 2 \rightarrow A,\ 3 \rightarrow D,\ 4 \rightarrow B$

This means that programmer 1 is assigned programme C, programmer 2 is assigned programme A, and so on. The minimum time taken in developing the programmes is

= 80 + 80 + 100 + 90 = 350 min.

**Example 2:** A company is producing a single product and selling it through five agencies situated in different cities. All of a sudden, there is a demand for the product in five more cities that do not have any agency of the company.

The company is faced with the problem of deciding on how to assign the existing agencies to dispatch the product to the additional cities in such a way that the travelling distance is minimised. The distances (in km) between the surplus and deficit cities are given in the following distance matrix.

| Deficit City / Surplus city | I | II | III | IV | V |
|---|---|---|---|---|---|
| **A** | 160 | 130 | 175 | 190 | 200 |
| **B** | 135 | 120 | 130 | 160 | 175 |
| **C** | 140 | 110 | 155 | 170 | 185 |
| **D** | 50 | 50 | 80 | 80 | 110 |
| **E** | 55 | 35 | 70 | 80 | 105 |

Determine the optimum assignment schedule.

**Solution:** Subtracting the minimum element of each row from every element of that row, we have

| | I | II | III | IV | V |
|---|---|---|---|---|---|
| A | 30 | 0 | 45 | 60 | 70 |
| B | 15 | 0 | 10 | 40 | 55 |
| C | 30 | 0 | 45 | 60 | 75 |
| D | 0 | 0 | 30 | 30 | 60 |
| E | 20 | 0 | 35 | 45 | 70 |

Subtracting the minimum element of each column from every element of that column, we have

| | I | II | III | IV | V |
|---|---|---|---|---|---|
| A | 30 | 0 | 35 | 30 | 15 |
| B | 15 | 0 | 0 | 10 | 0 |
| C | 30 | 0 | 35 | 30 | 20 |
| D | 0 | 0 | 20 | 0 | 5 |
| E | 20 | 0 | 25 | 15 | 15 |

We now assign zeroes by drawing rectangles around them as explained in Example 1.

Thus, we get

| | I | II | III | IV | V |
|---|---|---|---|---|---|
| A | 30 | $\boxed{0}$ | 35 | 30 | 15 |
| B | 15 | $\cancel{0}$ | $\boxed{0}$ | 10 | $\cancel{0}$ |
| C | 30 | $\cancel{0}$ | 35 | 30 | 20 |
| D | $\boxed{0}$ | $\cancel{0}$ | 20 | $\cancel{0}$ | 5 |
| E | 20 | $\boxed{\cancel{0}}$ | 25 | 15 | 15 |

Since the number of assignments is less than the number of rows (or columns), we proceed from Step 5 onwards of the Hungarian method as follows:

i) we tick mark ( √ ) the rows in which the assignment has not been made. These are the 3rd and 5th rows.

ii) we tick mark ( √ ) the columns which have zeroes in the marked rows. This is the 2nd column.

iii) we tick mark ( √ ) the rows which have assignments in marked columns. This is the 1st row.

iv) again we tick mark ( √ ) the column(s) which have zeroes in the newly marked row. This is the 2nd column, which has already been marked. There is no other such column. So, we have

| | I | II | III | IV | V | |
|---|---|---|---|---|---|---|
| A | 30 | $\boxed{0}$ | 35 | 30 | 15 | √ |
| B | 15 | $\cancel{0}$ | $\boxed{0}$ | 10 | $\cancel{0}$ | |
| C | 30 | $\cancel{0}$ | 35 | 30 | 20 | √ |
| D | $\boxed{0}$ | $\cancel{0}$ | 20 | $\cancel{0}$ | 5 | |
| E | 20 | $\cancel{0}$ | 25 | 15 | 15 | √ |
| | | √ | | | | |

We draw straight lines through unmarked rows and marked columns as follows:

| | I | II | III | IV | V | |
|---|---|---|---|---|---|---|
| A | 30 | $\boxed{0}$ | 35 | 30 | 15 | √ |
| B | 15 | $\cancel{0}$ | 0 | 10 | $\cancel{0}$ | |
| C | 30 | $\cancel{0}$ | 35 | 30 | 20 | √ |
| D | 0 | $\cancel{0}$ | 20 | $\cancel{0}$ | 5 | |
| E | 20 | $\cancel{0}$ | 25 | 15 | 15 | √ |
| | | √ | | | | |

We proceed as follows, as explained in step 6 of the Hungarian method:

i) We find the smallest element in the matrix not covered by any of the lines. It is 15 in this case.

ii) We subtract the number '15' from all the uncovered elements and add it to the elements at the intersection of the two lines.

iii) Other elements covered by the lines remain unchanged. Thus, we have

| | I | II | III | IV | V |
|---|---|---|---|---|---|
| A | 15 | 0 | 20 | 15 | 0 |
| B | 15 | 15 | 0 | 10 | 0 |
| C | 15 | 0 | 20 | 15 | 5 |
| D | 0 | 15 | 20 | 0 | 5 |
| E | 5 | 0 | 10 | 0 | 0 |

We repeat Steps 1 to 4 of the Hungarian method and obtain the following matrix:

| | I | II | III | IV | V |
|---|---|---|---|---|---|
| A | 15 | $\cancel{0}$ | 20 | 15 | $\boxed{0}$ |
| B | 15 | 15 | $\boxed{0}$ | 20 | $\cancel{0}$ |
| C | 15 | $\boxed{0}$ | 20 | 15 | 5 |
| D | $\boxed{0}$ | 15 | 20 | $\cancel{0}$ | 5 |
| E | 5 | $\cancel{0}$ | 10 | $\boxed{0}$ | 0 |

Since each row and each column of this matrix has one and only one assigned 0, we obtain the optimum assignment schedule as follows:

$$A \rightarrow V,\ B \rightarrow III,\ C \rightarrow II,\ D \rightarrow I,\ E \rightarrow IV$$

Thus, the minimum distance is 200 + 130 + 110 + 50 + 80 = 570 km.

You should pause here and try to solve the following assignment problem to check your understanding.